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ECONOMIC EQUILIBRIUM UNDER DEFORMATION OF THE ECONOMY.(U)

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by

10 Charles R. Engles

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ABSTRACT

A philosophical problem arises when one attempts to predict a competitive economy's response to a fundamental change in its structure with the aid of a competitive equilibrium model. Unless the model is known to admit unique solutions, there is little basis for assuming that the computed equilibrium will be attained, even though the model accurately describes the economy's structure and the behavior of its agents. If, however, one is able to arrive at the new model by continuously deforming the old one, then the two versions generally admit solutions which are connected by a path of equilibria arising from the continuum of intermediate economies. By ascribing a suitable dynamic interpretation to the deformation, one obtains a rationale for expecting the path-connected solutions to be mutually attained.

The description of economic deformations and the computation of equilibrium paths is the central theme of this study. A general mathematical framework for modeling economies under deformation is developed by expanding Herbert Scarf's original activity analysis formulation to include uncountable unit activity sets, unbounded multi-valued demand correspondences, and tax and revenue systems similar to those introduced by John Shoven and John Whalley. Deformations of virtually all economic constructs are allowed in this general model.

The computation of equilibrium paths is accomplished by a simplicial pivot algorithm designed along the lines of the homotopy-type fixed point techniques pioneered by Curtis Eaves. The dimension normally used to

refine piecewise linear approximations now serves as the index of the economic deformation. To make this approach viable in practice, a new family of triangulations of Euclidean space is fashioned out of two conventional triangulations invented by Michael Todd. The geometry of these triangulations can be dynamically altered by the algorithm as it attempts to maintain uniform approximation error along the equilibrium path.

The economic model and computational algorithm are translated into a set of computer routines which generate explicit numerical approximations to equilibrium paths for a variety of examples. Due to the vast amount of information embodied in an equilibrium path, problems of this type require a great deal of computational effort. A detailed analysis of the behavior of the algorithm on a series of test problems is presented in the final chapter.

PREFACE

The research embodied in this dissertation was conducted in 1974 and 1975 while the author was in residence at Stanford University. At that time certain references, which have since been published in journals and conference proceedings, were available in manuscript or technical report form [16], [17], [20], [21]. Although the techniques developed herein represented the state of the art in 1975, the author recognizes that advances in the area of fixed point calculation made during the past three years could perhaps be adapted to expand, simplify, or enhance the performance of these techniques.

A small group of individuals played significant personal roles in the evolution of this work. The author is grateful to Curtis Eaves for originally proposing this line of inquiry, for acquainting the author with leading authorities in the field, and for accepting the author as his friend as well as his student. A special debt of gratitude is owed to John Shoven who, despite an unusually demanding schedule, found time to meet frequently with the author to offer economic coaching, computational insight, and much-needed doses of optimism and encouragement. The author benefited throughout his years at Stanford from the wise and benevolent counsel of Richard Cottle. Romesh Saigal provided helpful suggestions on the design of the computer programs, and Gail Lemmond Stein, one of the area's premier mathematical typists, translated the manuscript into legible form.

The author's studies and research at Stanford were made possible by a National Science Foundation Graduate Fellowship. Access to the excellent computing facilities at the Stanford Linear Accelerator Center was provided by the Systems Optimization Laboratory under the direction of George Dantzig.

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CHAPTER 1

INTRODUCTION

1.1. Background of the Problem

A little over a decade after general competitive equilibria were first shown to exist, Herbert Scarf breathed new life into competitive equilibrium theory by developing a workable algorithm for computing equilibrium prices and commodity flows in general Walrasian models [12]. The algorithm grew out of Scarf's earlier work on the computation of fixed points of a continuous mapping. Like the techniques that preceded it, the algorithm derived its validity from the anti-cycling principle of Lemke and Howson. Scarf's approach transcended conventional fixed-point methods, however, by operating on a space half the dimension of the one normally encountered in the fixed-point step of existence proofs. This material reduction of dimension was achieved at the modest expense of requiring technology to exhibit constant returns to scale (CRS). The algorithm also harbored an ability to converge even when demand responses were unbounded. Hence the elaborate and non-constructive truncation arguments found in virtually all pure existence proofs became superfluous for CRS models.

Scarf's achievement raised for the first time the possibility of extending the scope of competitive equilibrium theory from the realm of pure theory into the empirical arena. The prospect of fitting the theory to reality, however, highlighted some of the more artificial aspects of the strict neo-classical interpretation. Many of these features will either have to be revised or abandoned if the theory is ever to meet the scientific criterion of providing sound predictions. A first step in this direction has been taken by two of Scarf's students, John Shoven and John Whalley, who incorporated certain aspects of government fiscal policy into a competitive equilibrium framework [19].

The present study abandons the requirement that all consumption and production decisions be made at one instant of time for the entire life-span of the economy. Instead the view is taken that a competitive economy evolves through a series of short-to-medium term responses by consumers and producers to longer term exogenous changes in the environment. These exogenous changes could result either from the conscious actions of governmental authorities or from autonomous factors such as technological innovation, shifts in consumer tastes, or unperceived depletion of a vital resource. They could be completely independent of economic behavior or linked to it in some specific manner, perhaps even stochastically. Based on this revised intertemporal interpretation, forecasts of future economic behavior can be prepared by first estimating likely values of the autonomous factors, then allowing for proposed government policy, and finally solving for a short-run equilibrium via Scarf's algorithm. This procedure parallels the intuitive approach taken by most economists when asked to predict the future value of some economic variable.

Unfortunately the procedure conceals a serious technical flaw. Even if the parameters of some future economy are known with complete certainty and the economy attains a competitive equilibrium consistent with these parameters, the computed forecast could still be wrong. This can occur whenever multiple equilibria are present. In such instances there is no way of knowing whether the equilibrium computed by Scarf's algorithm will be the one attained. The problem is compounded by the fact that no general method exists for locating all the equilibria in a given model. This predicament is especially frustrating if one wishes to design economic policy based on the forecasts.

The multiple equilibrium dilemma is illustrated in Figure 1.1.1. A hypothetical competitive economy occupies equilibrium state A at time t_0. Exogenous parameter changes transform the initial economy into one admitting three equilibria X, Y, and Z at time t_1. Which of the three states will actually be attained is open to question, however.

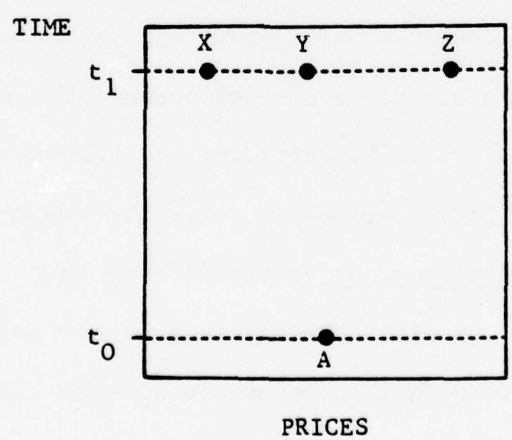


Figure 1.1.1. The multiple equilibrium dilemma.

The potential ambiguity of forecasts based on competitive equilibrium models has been recognized by mathematical economists for some time. Arrow and Hahn [2] conclude that "this problem must be intimately related to that of the uniqueness of an equilibrium and it is pretty clear that we shall not expect to get very far without stipulating one or the other of the conditions that ensure such uniqueness." The present study submits that this conclusion is unduly pessimistic and offers instead a method for obtaining unambiguous forecasts even in the presence of multiple equilibria.

The heart of the method consists of the notion of continuous deformation of a competitive economy. The underlying assumption is that the exogenous parameters which determine short-run equilibria in some economy evolve continuously over time. An alternative description of this process is that the economy undergoes a deformation. The end product of the deformation is a continuum of economies, one for each time point in some interval. Each intermediate economy presumably possesses its own set of competitive equilibria. Plotting the equilibrium set of each economy against its time index produces a revealing subset of price-index space called the <u>equilibrium graph</u> of the deformation. The geometry of the equilibrium graph provides the key to resolving ambiguity in forecasts.

A variety of geometric forms are possible. The equilibrium graph of the hypothetical economy discussed earlier could, for example, assume any of the shapes displayed in Figures 1.1.2 through 1.1.4. One feature must always be present, however: a connected subset of equilibria spanning the interval of the deformation. This phenomenon is intimately related to the fixed point theorem of Felix Browder.

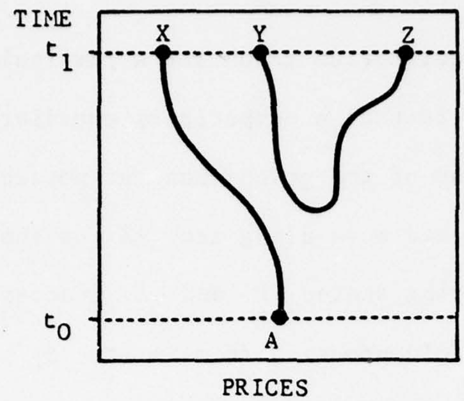


Figure 1.1.2. Mutually inaccessible sets of equilibria.

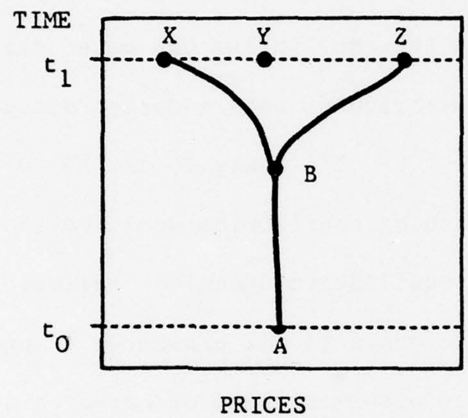


Figure 1.1.3. The divergence effect.

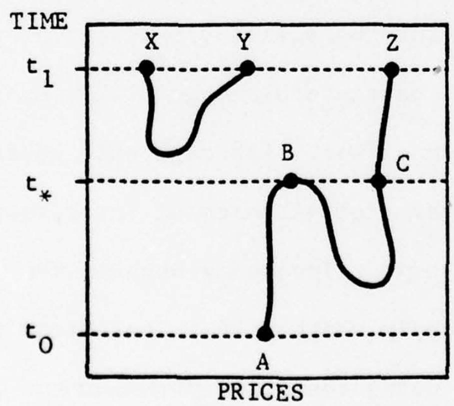


Figure 1.1.4. The catastrophe effect.

Given the equilibrium graph for a particular deformation, the operational hypothesis is that a competitive equilibrium always follows a connected component of the graph whenever possible. Thus in Figure 1.1.2 the equilibrium would move along arc AX as the time index advanced from t_0 to t_1, rendering states Y and Z inaccessible. In Figure 1.1.3 the equilibrium would progress from A to B, then veer to the right or the left depending on the detailed adjustment mechanics in operation at that instant. The situation in Figure 1.1.4 is somewhat more complex. At time t_*, after the equilibrium has moved from A to B, the economy would experience a period of severe market disruption while prices readjusted to state C. If, however, arc XY were to dip below the t_* level, then the ensuing equilibrium would be impossible to predict.

The view of equilibrium dynamics implicit in these examples is that the economy responds to all parameter changes by restoring equilibrium, and that it does so with a minimum of market dislocation. A rigorous defense of this interpretation would require the demonstration of some form of stability for the equilibria along the path and some assurance that the exogenous parameters change slowly enough to permit economic adjustments to take effect. Although such questions are interesting and perhaps necessary for purposes of interpretation, they are of secondary concern to this study, principally because the computational techniques developed herein apply whether such conditions are present or not.

The idea of using connected components of equilibrium graphs to resolve ambiguity in economic forecasts was inspired largely by the work of Curtis Eaves in the computation of fixed points [7], [9]. An extension of Eaves' methods also provides the means for computing numerical approximations to such components. Eaves originally set out to remedy a weakness in Scarf's general purpose fixed-point algorithm, namely that once an approximate solution is found, the location of that solution contributes nothing to the search for a more accurate solution. Eaves resolved this difficulty by introducing the topological concept of homotopy into the fixed-points arena. Simply stated, one appends an extra dimension to the domain of the problem of interest and uses this dimension to index a family of approximations to the original problem. A simplicial pivot algorithm follows solutions of the approximate problems closer and closer to a solution of the problem of interest. The family of approximate problems and the path of solutions are special cases, respectively, of the homotopy and connected set of fixed points that arise in Browder's theorem.

As soon as Eaves' algorithm became widely known, speculation arose as to whether the technique could be extended to compute Browder paths for more general types of homotopies. The only apparent requirement was to use the extra dimension to index an arbitrary family of problems rather than a series of approximations to a particular problem. It was further conjectured that the algorithm could be adapted to trace the evolution of a competitive equilibrium as the parameters defining the economy changed over time [4]. The present study realizes the ambitions of both conjectures by developing a workable algorithm for approximating connected components of equilibrium graphs for economies under deformation. At the same time the advantages of continuously refining grids are made available for single equilibrium calculations with CRS competitive models.

The approximate equilibrium graphs generated by the algorithm take the form of special polygonal paths. The accuracy of approximation along each path may be prescribed arbitrarily in advance. Examples of the types of paths that the algorithm would produce for the equilibrium graphs in Figures 1.1.2 and 1.1.4 are shown in Figures 1.1.5 and 1.1.6. For structures such as these the algorithm always charts the true course of the economy. Other geometries can, unfortunately, be more elusive. The equilibrium graph in Figure 1.1.3, for example, would cause the algorithm to arbitrarily select one of the two upper branches even though the economy might follow the other. Also, when the initial economy admits multiple equilibria, the algorithm could conceivably follow a path which misses the empirically observed equilibrium altogether. Intricate geometries notwithstanding, the algorithm can successfully resolve the potential ambiguity of forecasts for a wide class of interesting examples, not the least of which are those admitting unique equilibria but which are not known to do so.

1.2. Scope of the Study

The computational procedure developed in this study was designed to handle a very general class of CRS Walrasian models. Many types of deformations can be applied to these models, including changes in consumer tastes and wealth, production technology, resource availability, and taxes and tariffs. To permit the latter type of displacement, tax and revenue systems of the form introduced by Shoven and Whalley [19] are

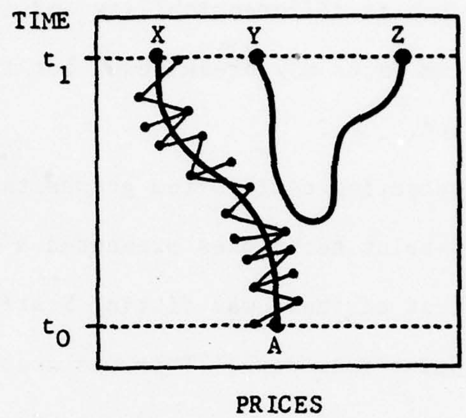


Figure 1.1.5. Polygonal path approximation to equilibrium graph of Figure 1.1.2.

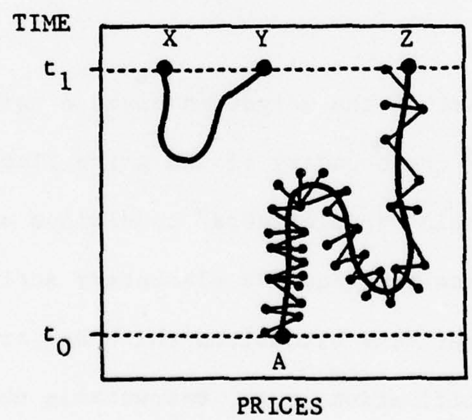


Figure 1.1.6. Polygonal path approximation to equilibrium graph of Figure 1.1.4.

built into the model. The only regularity conditions imposed on the deformation and the economies are continuity and its extensions to correspondences, namely upper and lower semi-continuity. Without additional properties such as differentiability and stability, the interpretation of equilibrium paths may break down, but the mechanics of computing them are not affected.

The idea of computing equilibrium graphs through an extension of homotopy-type fixed-point techniques presented a number of technical challenges. The first of these was fitting Scarf-type economic labels onto Eaves' fixed-point framework. This was accomplished in part by ascribing an extra degree of range freedom to Eaves' abstract labeling. The extra freedom in turn necessitated an additional assumption to insure that certain linear inequality systems remained bounded. To accommodate parametric change in the economies an extra degree of domain freedom was also added to the labeling. These modest generalizations are carried out in Chapter 2.

The other part of the merger involved a refinement of Scarf's method of labeling the boundary of the price simplex. This was necessary in order that certain "completeness" conditions of the abstract algorithm be met. At the same time Scarf's elementary activity analysis model was generalized to cover some situations which had arisen in practice but had no formal justification, e.g., uncountable unit activity sets and unbounded multi-valued demand functions. The ability to handle unbounded demand functions is particularly important since truncation of the type employed in most existence proofs cannot be performed numerically. As a result of these extensions, CRS competitive models of full "existence proof" generality can now be solved numerically. Beyond the enhancement of Scarf's model in these traditional directions, tax and revenue systems à la Shoven and Whalley were included to permit the evaluation of fiscal policy. Finally, the context of the model was broadened from a single economy to a continuous family of economies, each possessing the same structural components but different parametric values. A comprehensive treatment of the economic model and labeling appears in Chapter 3, along with proofs that the algorithm clusters in the limit around a connected component of the equilibrium graph, and that after a finite number of iterations a meaningful approximation of pre-determined accuracy is available.

Once the union of Scarf's economic labeling and Eaves' fixed-point scheme was consummated, a thorny practical matter still had to be resolved. The problem was that none of the conventional triangulations used in homotopy-type fixed-point algorithms were suitable for computing equilibrium graphs, because they all led to grossly uneven quality of approximation along the graph. An even more disturbing realization was that no single triangulation could provide the uniform quality desired for all problems. A way out of this predicament was found through the dynamic manifold definition principle expounded in Chapter 4. Using this principle two new families of triangulations were constructed from portions of Michael Todd's J_1 and J_3 triangulations [20]. Every example submitted to the algorithm automatically causes a triangulation from one of these families to be custom tailored to its accuracy needs.

In order to test the efficacy of the algorithm, a series of numerical experiments was conducted using computer programs designed to implement the procedure. The experiments consisted of thirteen test problems, each of which fit one of the specialized versions (developed in Chapter 5) of the general economic model. The results of the experiments are reported and analyzed in detail in Chapter 6. The experiments demonstrate conclusively that the algorithm functions as intended but expends large amounts of computational effort. An analysis of iteration counts suggests that the effort results not so much from the inefficiency of the algorithm as from the vast amount of information inherent in the extremely precise approximate equilibrium graphs that were generated. Owing to this inherent expense, applications presently appear to be limited to models with a dozen or so commodities when high precision is required. Relaxing accuracy requirements by a few percent would permit twenty commodity examples to be solved in a reasonable amount of time. Both of these ceilings will, of course, rise as computer technology advances.

The development of the techniques presented in this study would not have been possible without the previous accomplishments of five men. Their influence ranges from the conceptual plane to specific formalisms and proofs. The economic labeling and general logic of the convergence and finite approximation proofs are due to Herbert Scarf. The deformation concept and elegant formalism of Chapter 2 are due to Curtis Eaves. John Shoven and John Whalley influenced the study in several ways: their method of adding taxes to competitive equilibrium models was copied almost verbatim; much of the data in the numerical experiments was supplied by Shoven; and their pioneering efforts in the empirical comparison of equilibria raised the fundamental questions that motivated the study in the first place. Finally, the algorithm could never have been made computationally feasible without Michael Todd's "union jack" triangulations [20], and without his theoretical measures of directional density [21], the efficiency of the procedure would have been difficult to judge.

Although this study is oriented exclusively toward economic equilibrium calculations, many of its techniques can be adapted to the computation of general parametric fixed points. The relevant parts for this purpose are the fundamental algorithm of Chapter 2 and the dynamically defined manifolds and control heuristics of Chapter 4. Parametric fixed-point problems may actually be easier to solve than the economic models considered here because of greater regularity in the labeling.

1.3. Notation and Conventions

The terminology and elementary mathematical tools used in this study are hybrids of those found in the literatures of mathematical economics and operations research. The influence of the latter field is apparent in the heavy use of vector and matrix notation. The main purpose of this section is to explain the symbols, conventions, etc., which differ in some respect from standard usage.

The general setting of the study is $(n+1)$-dimensional Euclidean space R^{n+1}, where $n \geq 0$. The axes of R^{n+1} are indexed $0, 1, \ldots, n$. Vectors in R^{n+1} are denoted by lower case Greek and Roman letters. No notational distinction is made between row and column vectors, but the general rule applies throughout that all vectors are column vectors unless they pre-multiply a matrix or another vector.

The components of a vector x in R^{n+1} are denoted $x(i)$ for $0 \leq i \leq n$. If α is a non-empty subset of $\{0, \ldots, n\}$ with $|\alpha|$ members, then $x(\alpha)$ denotes the vector in $R^{|\alpha|}$ whose components are $x(i)$ for $i \in \alpha$. A subscript on a vector or any other object merely distinguishes that object from others denoted by the same symbol. A superscript on a vector or any other object indicates the position of the object in a sequence. Thus the symbol $x_j^k(i)$ denotes the i-th component of the k-th term of the j-th sequence of x's. A single exception to this rule occurs in Chapter 5 where the continuous parameter t appears as a superscript.

Vectors in the canonical basis $\mathcal{J}_{n+1}$ of R^{n+1} are denoted e_j for $0 \leq j \leq n$. Whenever R^{n+1} is factored into components R^{m+1} and R^{n-m}, canonical vectors for these subspaces will be denoted by the symbols

f and g respectively, with no subscripts to distinguish along which dimension each vector lies. The identity matrix of any dimension will be denoted by the letter I. The letter e will denote a vector all of whose components are 1 and whose dimension is determined by context. Pre-multiplying a vector by e merely sums the components of that vector.

Three order relations in R^{n+1} are used in subsequent chapters. For x, y in R^{n+1},

(a) $x \leq y$ means $x(i) \leq y(i)$ for all coordinates $0 \leq i \leq n$;

(b) $x < y$ means $x \leq y$ but $x \neq y$;

(c) $x \ll y$ means $x(i) < y(i)$ for all coordinates $0 \leq i \leq n$.

If $x = 0$, then y is said to be non-negative, semi-positive, or strictly positive according to (a), (b), or (c).

The positive and negative parts x^+ and x^- of a vector x in R^{n+1} are defined by

(a) $$x^+(i) = \begin{cases} x(i) , & \text{if } x(i) > 0 \\ 0 , & \text{if } x(i) \leq 0 \end{cases}$$

(b) $$x^-(i) = \begin{cases} -x(i) , & \text{if } x(i) < 0 \\ 0 , & \text{if } x(i) \geq 0 \end{cases}$$

where $0 \leq i \leq n$.

The sign of a vector x in R^{n+1} is defined by

$$(\operatorname{sgn} x)(i) = \begin{cases} 1 , & \text{if } x(i) > 0 \\ 0 , & \text{if } x(i) = 0 \\ -1 , & \text{if } x(i) < 0 \end{cases}$$

for $0 \leq i \leq n$.

All algebraic operations and order relations in R^{n+1} have natural extensions to the power set of R^{n+1}. If the symbol $*$ denotes addition, subtraction, or inner product and $A, B \subset R^{n+1}$, then $A * B$ is the set of all objects $a * b$ where $(a,b) \in A \times B$. Similarly if $*$ denotes one of the order relations in R^{n+1}, then $A * B$ is true iff $a * b$ holds for each (a,b) in $A \times B$.

Some additional operations on subsets A, B of R^{n+1} include:

(a) A_+ - the non-negative vectors in A;

(b) card A - the cardinality of A;

(c) conv A - the convex hull of A;

(d) aff A - the affine hull of A;

(e) pos A - the convex cone generated by A;

(f) $A \backslash B$ - the set theoretic difference of A and B.

Convex polyhedra are sets formed by intersecting a finite number of closed halfspaces of R^{n+1}. The facets of a convex polyhedron are the maximal convex subsets of the relative boundary of the polyhedron. A special class of convex polyhedra used extensively in Chapter 4 is the class of j-dimensional simplices for $0 \leq j \leq n$. A j-dimensional simplex σ is the convex hull of $j+1$ affinely independent points $x_0, \ldots, x_j$ called its vertices. Such a simplex σ is denoted by the $(j+1)$-tuple $(x_0, \ldots, x_j)$, which implies an ordering of the vertices. A face of σ is the relative interior of a simplex determined by a (possibly improper) subset of its vertices. (Note that faces are relatively open while facets are relatively closed.)

A collection of $(n+1)$-simplices constitutes a triangulation of some subset of R^{n+1} if the faces of all simplices in the collection partition the given subset.

The standard n-dimensional simplex $(e_0, \ldots, e_n)$ will be denoted by the letter S, or if the value of n needs to be made explicit, by the symbol S_n. If T is an interval, then facets of the product set $S \times T$ which are extensions of facets of S are denoted by F_i, where i is the unused dimension of R^{n+1}.

The set A^{n+1} consists of all vectors in R^{n+1} whose components are ± 1. The symbol Ψ_{n+1} denotes the group of permutations on $\{0, 1, \ldots, n\}$.

If σ is a finite ordered subset of R^k consisting of n elements $v_0, \ldots, v_{n-1}$ and L is any mapping from R^k to R^m, then $L(\sigma)$ is the $(m\times n)$-matrix whose columns are $L(v_0), \ldots, L(v_{n-1})$. The collection of all $m\times n$ real matrices is denoted by $R^{m\times n}$. Elements of a matrix A in $R^{m\times n}$ are denoted by $A(i,j)$, rows by $A(i,\cdot)$, and columns by $A(\cdot,j)$.

Lexicographic linear inequality systems are needed in the development of the fundamental algorithm in Chapter 2. The usual lexicographic ordering $\underline{\succ}$ in R^n is extended row-wise to $R^{m\times n}$ in the same way that the usual non-negative ordering $\geq$ in R^1 is extended to R^m. A concise account of the fundamentals of lexicographic linear inequality systems appears in Section 1.2 of [6].

A metric topology is induced on R^{n+1} by one of three equivalent norms:

(a) $\|x\|_\infty = \max_i |x(i)|$;

(b) $\|x\|_1 = |x(0)| + \cdots + |x(n)|$;

(c) $\|x\|_2 = (x(0)^2 + \cdots + x(n)^2)^{1/2}$.

Observe that for x in R^{n+1}, $\|x\|_\infty \le \|x\|_2 \le \|x\|_1$. Also note that if x is partitioned into (x_1, x_2), then $\|x\|_1 = \|x_1\|_1 + \|x_2\|_1$. Hölder's inequality $|xy| \le \|x\|_\infty \|y\|_1$ is used repeatedly in Section 3.5.

If A, B are subsets of R^{n+1} and $\| \ \|_p$ is one of the norms defined above, then $\mathrm{dist}_p (A,B) = \inf\{\|a-b\|_p : (a,b) \in A \times B\}$. The definition extends naturally to the case where either A or B is a point. Similarly define $\mathrm{diam}_p\, A = \sup\{\|a-b\|_p : a,b \in A\}$.

Whenever a discussion involving norms, distances, or diameters is insensitive to which norm is used, the subscript on $\| \ \|_p$, dist_p, or diam_p will be suppressed.

The symbol $\langle \sigma^k \rangle$ represents an infinite sequence of objects σ^k, where k implicitly ranges through the set of non-negative integers Z_+. If $\langle \sigma^k \rangle$ is a sequence of subsets of R^{n+1} and $x \in R^{n+1}$, then $\sigma^k \to x$ means that $\mathrm{diam}(\sigma^k \cup \{x\}) \to 0$ as $k \to \infty$. If $n = 0$ then $\sigma^k \to +\infty$ means that σ^k eventually leaves every interval $(-\infty, N]$ for N in Z_+.

Much of the analysis in this study is conducted with correspondences, i.e., mappings from R^ℓ to $(R^{n+1})^*$, the collection of non-empty subsets of R^{n+1}. Two regularity concepts generalize the usual notion of continuity to correspondences:

(a) Upper semi-continuity: A correspondence $\Phi : R^\ell \to (R^{n+1})^*$ is said to be upper semi-continuous (u.s.c.) iff $x^k \to x \in R^\ell$, $y^k \in \Phi(x^k)$, and $y^k \to y \in R^{n+1}$ jointly imply that $y \in \Phi(x)$. The upper semi-continuity of a correspondence is equivalent to the correspondence having a closed graph.

(b) Lower semi-continuity: A correspondence $\Phi : R^{\ell} \to (R^{n+1})^*$ is said to be lower semi-continuous (l.s.c.) iff $x^k \to x \in R^{\ell}$ and $y \in \Phi(x)$ jointly imply that $\exists\ y^k \in L(x^k)$ s.t. $y^k \to y$.

A correspondence which is both u.s.c. and l.s.c. is said to be continuous. The notion of uniform continuity is extended to correspondences in Lemma A.5 and Definition A.6. The reader should examine A.6 before reading the proof of Theorem 3.5.4.

A concept from metric space topology called the Lebesgue number of a covering is needed at two or three points in the analysis. If $\{U_\alpha\}_{\alpha \in \mathcal{a}}$ is an open covering of a compact metric space X, then there exists a $\delta > 0$ such that any subset A of X whose diameter is less than δ lies in some U_α. Any such constant δ is called a Lebesgue number of the covering $\{U_\alpha\}_{\alpha \in \mathcal{a}}$.

Chapter 3 contains many integer intervals of the form $I+1 \leq j \leq J$. Whenever $I = J$, this interval is defined to be the empty set $\emptyset$. In this case any summation indexed by the interval, such as $\sum_{j=I+1}^{J} y(j)$, is defined to be zero.

The term "algorithm" is used in this study to describe any iterative computational procedure, whether or not the procedure terminates after a finite number of steps.

The body of this dissertation is organized into chapters, sections, and items within sections. Three levels of indexing are used to keep track of these entities, e.g., 3.2.5 refers to item 5 in Section 2 of Chapter 3. Figures and tables are indexed independently of the other items in each section. An exception to these rules is Appendix A, which contains no sections.

References are designated by enclosing the entry number in square brackets, e.g., [12].

CHAPTER 2

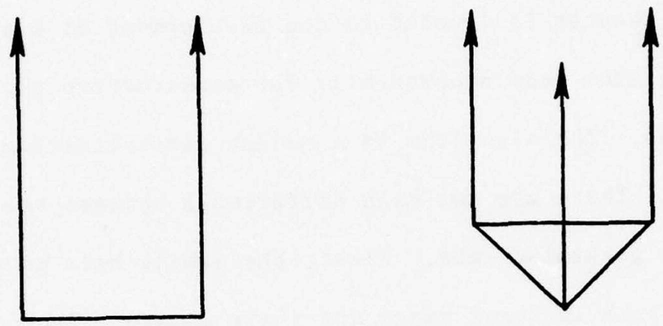
THE FUNDAMENTAL ALGORITHM

This chapter is devoted to the development of a special type of search algorithm used subsequently for constructive proofs and for actual calculations. The algorithm is a modest generalization of the one appearing in [8]. There are two main differences between the algorithm of [8] and the one presented here. First, the labels here have an extra degree of freedom both in their range and their domain. Second, the labels here form a convex cone containing an arbitrary vector rather than a convex hull containing the origin. The extra degree of range freedom necessitates an additional assumption on the labeling and the pseudomanifold to insure boundedness of the linear inequality systems formed by the labels.

Roughly speaking the algorithm steps through a special type of grid called a pseudomanifold. The path that the algorithm follows is determined by vector-valued labels attached to grid points. The labels and the grid points contain the information needed to construct an approximate solution to some underlying problem. In subsequent chapters a portion of the path generated by the algorithm will be used as an approximation to some equilibrium graph.

2.1. Labelings of $S \times [0,\infty)$.

The geometric setting for the fundamental algorithm and all subsequent theoretical and computational work is the cylinder $S \times [0,\infty)$. The factor S will house relative commodity prices and revenue flows, and the factor $[0,\infty)$ a function of the economy index. Points in $S \times [0,\infty)$ are denoted interchangably by the letter v and the pair (s,t) where $s \in S$ and $t \in [0,\infty)$. Figure 2.1.1 depicts two versions of $S \times [0,\infty)$.



$S_1 \times [0,\infty)$ $S_2 \times [0,\infty)$

Figure 2.1.1

Note that $S \times [0,\infty)$ has $n+2$ facets, namely the base $S \times \{0\}$ and the $n+1$ walls F_i for $i = 0, 1, \ldots, n$. The projection of $S \times [0,\infty)$ onto the $[0,\infty)$ factor is denoted by p_2.

2.1.1. DEFINITION. An arbitrary mapping $L : S \times [0,\infty) \to R^{n+1}$ is called a labeling of the cylinder $S \times [0,\infty)$.

Let L be a labeling of $S \times [0,\infty)$ and p be an arbitrary vector in R^{n+1}.

2.1.2. DEFINITION. A set $C \subset S \times [0,\infty)$ is said to be (L,p)-complete iff $p \in \text{pos}\, L(C)$.

2.1.3. DEFINITION. The pair (L,p) is defined to be proper iff

(a) the vertex set of $S \times \{0\}$ is (L,p)-complete;

(b) no facet of $S \times [0,\infty)$ other than $S \times \{0\}$ is (L,p)-complete.

The (L,p)-completeness property serves as the steering mechanism for the fundamental algorithm. Beginning with $S \times \{0\}$, the algorithm generates an infinite sequence of (L,p)-complete subsets of $S \times [0,\infty)$ by stepping through the grid structure defined in the next section.

2.2. Abstract Pseudomanifolds on $S \times [0,\infty)$

The fundamental algorithm like most general purpose fixed point algorithms operates on a special type of grid over the domain of interest. Such grids are variously known as triangulations, simplicial subdivisions, or simplicial complexes. An algebraic generalization of these structures specially tailored to the needs of the fundamental algorithm is defined below.

Let K^{n+1} be a collection of subsets of $S \times [0,\infty)$ of cardinality $n+2$. For $i = 0$ and $i = n$ let $K^i = \{\sigma : \sigma \subset \tau \in K^{n+1}$ and $\text{card}\ \sigma = i+1\}$. Elements of K^0, K^n and K^{n+1} are called abstract vertices, abstract n-simplices, and abstract (n+1)-simplicies respectively. If $\sigma \subset \tau$ where $\sigma \in K^n$ and $\tau \in K^{n+1}$, then σ is called an abstract facet of τ.

2.2.1. DEFINITION. K^{n+1} is defined to be an abstract pseudomanifold on the cylinder $S \times [0,\infty)$ iff the following four conditions hold:

(a) The vertex set of $S \times \{0\}$, denoted σ^0, belongs to K^n and is the only n-simplex of K^n contained in $S \times \{0\}$;

(b) Each σ in K^n is a facet of precisely one or two τ in K^{n+1};

(c) A simplex σ in K^n is a facet of precisley one τ in K^{n+1} iff σ is contained in a facet of $S \times [0,\infty)$;

(d) For each t in $[0,\infty)$ only finitely many σ in K^n meet $S \times [0,t]$.

A simple example of an abstract pseudomanifold on $S_1 \times [0,\infty)$ appears in Figure 2.2.1. The vertex set of each triangle yields one $(n+1)$-simplex, the endpoints of each side of a triangle yield an n-simplex, and each vertex of a triangle constitutes an abstract vertex.

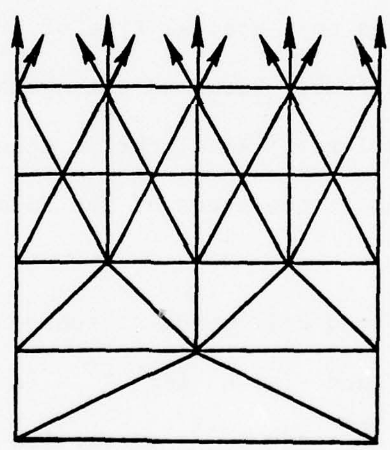


Figure 2.2.1. Abstract pseudomanifold on $S_1 \times [0,\infty)$

Beginning with σ^0 the fundamental algorithm generates a sequence of (L,p)-complete n-simplices in K^n. Given any σ^k in the sequence, σ^{k+1} is formed by replacing a vertex of σ^k with a vertex taken from one of the $(n+1)$-simplices containing σ^k. Condition (a) above allows the algorithm to start from σ^0, conditions (b) and (c) keep the algorithm going, and condition (d) insures that the algorithm will eventually climb arbitrarily high in the cylinder.

2.3. Very Complete Simplices and Adjacency

Only one ingredient in the fundamental algorithm remains to be specified, namely how to select a vertex of σ^k for replacement when σ^k is transformed into σ^{k+1}. The selection is determined by a linear inequality system involving labels on the vertices of σ^k. The condition that must ultimately be satisfied is that σ^{k+1} be (L,p)-complete. Unfortunately this condition alone will not identify a unique dropping vertex of σ^k, and without uniqueness the algorithm might cycle. Furthermore, some condition on the labeling L is required to insure that the linear inequality systems used for vertex selection are bounded. The first difficulty is overcome by an extension of the notion of (L,p)-completeness to lexicographic inequality systems. The second difficulty requires an additional assumption.

2.3.1. DEFINITION. An n-simplex σ in K^n is said to be (L,p)-very complete iff the lexicographic linear inequality system $L(\sigma)Y = [p,I]$, $Y \geq 0$ has a solution.

2.3.2. ASSUMPTION. (L,p) is a proper pair. (This and all other assumptions remain in effect throughout the current section.)

2.3.3. LEMMA. σ^0 is (L,p)-very complete.

Proof: The system $L(\sigma^0)y = p$, $y \geq 0$ has a solution y^0 according to 2.1.3(a). If the columns of $L(\sigma^0)$ were linearly dependent, the above system would possess a solution y with at least one zero component. By 2.1.3(b) this cannot occur, so $L(\sigma^0)^{-1}$ exists and $y^0 = L(\sigma^0)^{-1}p \gg 0$. Hence $Y^0 \equiv [L(\sigma^0)^{-1}p, L(\sigma^0)^{-1}] \geq 0$ and $L(\sigma^0)\, Y^0 = [p,I]$. □

2.3.4. ASSUMPTION. For each τ in K^{n+1} the linear inequality system $L(\tau)y = p$, $y \geq 0$ is bounded.

2.3.5. LEMMA. Each τ in K^{n+1} has either zero or two (L,p)-very complete facets.

Proof: Write $\tau = \{v_0, \ldots, v_{n+1}\}$. Consider the system

$$L(\tau)Y = [p,I]\ , \qquad Y \geq 0 \tag{2.3.6}$$

where $Y \in R^{(n+2)\times(n+2)}$. A facet $\sigma = \tau\setminus\{v_j\}$ is (L,p)-very complete iff 2.3.6 has a solution which does not use the j-th column of $L(\tau)$. Since $[p,I]$ has full row rank, any solution to 2.3.6 must use at least n+1 linearly independent columns of $L(\tau)$. Hence there is a 1-1 correspondence between (L,p)-very complete facets of τ and feasible bases of 2.3.6.

Given a feasible basis of 2.3.6, another feasible basis may be constructed by lexicographically pivoting on the unused column of $L(\tau)$. Such an operation will drive a column from the old basis because of Assumption 2.3.4, and the leaving column will be unique since $n+1$ columns are used in every solution of 2.3.6. Clearly the old basis and new basis are the only feasible bases of 2.3.6.

The proof is completed by observing that 2.3.6 is either infeasible or has a feasible basis. □

2.3.7. DEFINITION. Two distinct n-simplices of K^n are said to be adjacent iff they are both facets of some $(n+1)$-simplex in K^{n+1}.

2.3.8. LEMMA. σ^0 is adjacent to exactly one (L,p)-very complete n-simplex in K^n. Every other (L,p)-very complete simplex in K^n is adjacent to exactly two (L,p)-very complete simplices in K^n.

<u>Proof</u>: By 2.2.1(a) σ^0 belongs to K^n, and by 2.2.1(c) σ^0 is a facet of precisely one $(n+1)$-simplex τ^0 in K^{n+1}. By Lemma 2.3.3 σ^0 is (L,p)-very complete, hence by Lemma 2.3.5 τ^0 contains exactly one other (L,p)-very complete facet.

Now suppose σ is an (L,p)-very complete simplex distinct from σ^0. By 2.2.1(a) σ does not lie in $S \times \{0\}$, and by 2.1.3(b) σ does not lie in any other facet of $S \times [0,\infty)$. Hence by 2.2.1(b) and 2.2.1(c)

σ is contained in precisely two $(n+1)$-simplices of K^{n+1}. By Lemma 2.3.5 each of these $(n+1)$-simplices has one (L,p)-very complete facet other than σ. $\square$

2.4. The Algorithm

The machinery has now been assembled to formally state the fundamental algorithm. The statement consists of a starting n-simplex and an induction principle for generating successive n-simplices. An argument is then required to guarantee that the algorithm cannot cycle. The anti-cycling argument is based on the well-known Lemke-Howson graph principle.

Let (L,p) be a proper pair and let K^{n+1} be an abstract pseudo-manifold on $S \times [0,\infty)$ with facets K^n and vertices K^0. Assume (L,p) and K^{n+1} jointly satisfy 2.3.4. Define a sequence $\langle\sigma^k\rangle$ of adjacent (L,p)-very complete n-simpices of K^n as follows:

$k=0$: Let σ^0 be the vertex set of $S \times [0,\infty)$. σ^0 is (L,p)-very complete by Lemma 2.3.3.

$k=1$: Let σ^1 be the unique (L,p)-very complete simplex of K^n adjacent to σ^0. Such a σ^1 exists by Lemma 2.3.8.

$k\geq 2$: Suppose $\sigma^0, \sigma^1, \ldots, \sigma^{k-1}$ have been specified and are adjacent and (L,p)-very complete. By Lemma 2.3.8 σ^{k-1} is adjacent to two (L,p)-very complete simplices of K^n. One of these is σ^{k-2}. Define σ^k to be the other.

2.4.1. THEOREM. The sequence $\langle\sigma^k\rangle$ is distinct.

Proof: Clearly the finite sequence $\sigma^0, \ldots, \sigma^{k-1}$ is distinct for $k = 2$. Suppose it is distinct for some $k > 2$, but that σ^k coincides with one of the σ^i for $0 \leq i \leq k-1$. The definition of σ^k implies that σ^{k-1} is adjacent to σ^i and that $i < k-2$. Since σ^1 is the only (L,p)-very complete simplex adjacent to σ^0 and $k-1 > 1$, it follows that $i > 0$. But if $1 \leq i \leq k-3$, then σ^i is adjacent to the distinct n-simplices σ^{i-1}, σ^{i+1}, and σ^{k-1}, contradicting Lemma 2.3.8. The theorem follows by induction. □

2.4.2. COROLLARY. The projection of $\langle\sigma^k\rangle$ onto $[0,\infty)$ diverges to $+\infty$.

Proof: In view of the preceding theorem and 2.2.1(d), the algorithm must eventually vacate every truncated cylinder $S \times [0,t]$. Hence $p_2(\sigma^k) \to +\infty$ as $k \to \infty$. □

A possible realization of the fundamental algorithm in the pseudo-manifold of Figure 2.2.1 appears in Figure 2.4.1. For each 1-simplex σ^k, the succeeding 1-simplex σ^{k+1} is constructed by adding to σ^k the vertex opposite σ^k in the new triangle containing σ^k, and then by dropping an old vertex from σ^k. If S is regarded as a price simplex and t as an economy index, then a portion of the path swept out by $\langle\sigma^k\rangle$ will be used to approximate the equilibrium graph of a family of economies.

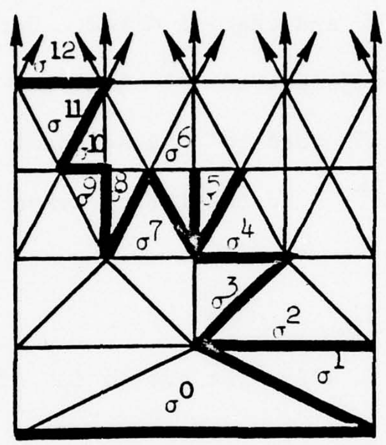


Figure 2.4.1. The fundamental algorithm in $S_1 \times [0,\infty)$

CHAPTER 3

THE ECONOMIC MODEL

This chapter introduces and analyzes the indexed family of CRS competitive economies which serves as the basic economic model of the study. The family extends Scarf's original Walrasian model [13] in several ways. The individual economies, for example, admit unbounded multi-valued demand correspondences, uncountable production activity sets, and multi-level tax systems of the form introduced by Shoven and Whalley [19]. More importantly, the economies are linked together into a continuum in which every economy is a deformed version of every other one.

The analysis of the economic model progresses through four stages of development. First, the components of the indexed family of economies are defined and discussed. Next, a minimal set of assumptions is introduced so that the fundamental algorithm of Chapter 2 can be adapted to the model. Then, under stronger but more economically meaningful assumptions, the path generated by the algorithm is shown to cluster around a connected component of the equilibrium graph of the family of economies. Finally, with the aid of two additional assumptions, a finite approximation theorem is established.

3.1. Components of the Model

The basic economic model consists of a family $\{\mathcal{E}(t)\}_{t\in T}$ of CRS Walrasian economies indexed by a real interval T. Generally T will denote the unit interval $[0,1]$ or the halfline $[0,\infty)$. Each economy $\mathcal{E}(t)$ in the family possesses the standard attributes of a general competitive economy, and in addition contains a number of revenue collection and distribution systems of the form introduced by Shoven and Whalley. The results of Shoven and Whalley are extended to a potentially wider class of revenue functions and to computation with multiple revenue systems. Although non-tax realizations of the revenue systems are conceivable (e.g., dividend distribution), a tax interpretation will be maintained here.

Three types of agents participate in each economy -- consumers, producers, and revenue handling agents. Consumers sell their labor and resource holdings, and purchase goods and services in such a way as to maximize satisfaction subject to the restriction that expenditures plus tax payments must not exceed endowment income plus revenue transfers. Producers purchase labor and raw materials and sell finished goods and services in such a manner as to maximize after tax profits. Revenue handling agents (usually government authorities) collect taxes from producers and consumers and redistribute the revenue among consumer groups. Since real governments spend money as well as collect it and give it away, they are often modeled both as consumers and as revenue handling agents.

The detailed characteristics of the economic agents are suppressed in this chapter, and instead their behavior is summarized in terms of market aggregates. This approach permits more generality and a neater mathematical development. Examples of consumption sets, utility functions, individual endowments, tax rates, etc., which lead to the hypothesized market aggregates will be presented in Chapter 5.

Every economy in the family $\{\mathcal{E}(t)\}_{t\in T}$ contains $m+1$ commodities indexed $0, \ldots, m$ $(m \geq 0)$ and $n-m$ revenue systems indexed $m+1, \ldots, n$

$(n > m)$. The commodities are traded at prices π in R_+^{m+1}, and the revenue systems are operated at levels r in R_+^{n-m}. Because all agent responses are required to be positively homogeneous of degree zero in prices and revenue levels, these parameters may be normalized so that $(\pi, r) \in S$. The family of economies may thus be considered to operate on $S \times T$. Points in this cylinder will be denoted as $v = (s,t) = (\pi,r,t)$ where $s = (\pi,r) \in S$ and $t \in T$. The t coordinate in (π,r,t) designates the economy $\mathcal{E}(t)$ to which (π,r) corresponds.

The formal specification of $\{\mathcal{E}(t)\}_{t\in T}$ is completed by assigning to each economy $\mathcal{E}(t)$ the following five components:

(a) A vector $w(t)$ in R^{m+1} of aggregate initial endowments. This vector is the sum of all consumers' initial commodity holdings. Positive components of $w(t)$ correspond to surpluses and negative components to deficits, hence the net market value of $w(t)$ at prices π is $\pi w(t)$.

(b) A market demand correspondence $\Xi(\cdot,t) : S \to (R^{m+1})^*$, positively homogeneous of degree zero in (π,r). This correspondence expresses total consumer demand for all commodities at prevailing prices π and revenue levels r. It is presumed defined even when some or all prices are zero, although such values need not depict actual consumer behavior (see Section 5.1). The dependence of Ξ on r reflects the influence of revenue transfers on consumer purchase decisions. Positive components of ξ in $\Xi(\pi,r,t)$ correspond to commodity purchases and negative components to sales, hence the cost of ξ at prices π is $\pi\xi$.

(c) A non-empty subset $\mathcal{B}(t)$ of R^{m+1} containing non-slack unit production activities. Vectors b in $\mathcal{B}(t)$ indicate technically feasible input-output combinations. Each producer in the economy owns and operates a subset of the activities in $\mathcal{B}(t)$, but since all production is CRS, the supposedly independent producers behave as if they were one consolidated producer. Positive components of b correspond to outputs and negative components to inputs, hence given prices π, the before-tax profit earned from operating b at unit level with market prices π is πb.

In addition to $\mathcal{B}(t)$ each economy is assumed to have available $m+1$ unit disposal activities $-\mathcal{J}_{m+1}$. Hence the total set of unit production activities available to economy $\mathcal{E}(t)$ is $\mathcal{A}(t) = \mathcal{B}(t) \cup (-\mathcal{J}_{m+1})$. Feasible production plans are constructed by selecting a number of unit activities from $\mathcal{A}(t)$ and operating them at non-negative levels. Hence *the set of feasible production plans for* $\mathcal{E}(t)$ is the convex cone pos $\mathcal{A}(t)$. Any production plan β in pos $\mathcal{A}(t)$ may be expressed as

$$\beta = [-f_0 \cdots -f_M \; b_{M+1} \cdots b_N]y \quad , \tag{3.1.1}$$

where $N \geq 0$, $-1 \leq M \leq N$, $y \in R_+^{N+1}$, $f_j \in \mathcal{J}_{m+1}$ for $0 \leq j \leq M$, and $b_j \in \mathcal{B}(t)$ for $M+1 \leq j \leq N$. The before-tax profit realized from executing plan β at market prices π is

$$\pi\beta = - \sum_{j=0}^{M} \pi f_j \, y(j) + \sum_{j=M+1}^{N} \pi b_j \, y(j) \; .$$

(d) A consumer tax function

$$\phi(\cdot,\cdot,t) : \bigcup_{(\pi,r)\in S} \Xi(\pi,r,t) \times \{(\pi,r)\} \to R^{n-m} .$$

For each price-revenue pair (π,r) in S and each consumption pattern ξ in $\Xi(\pi,r,t)$, $\phi(\xi,\pi,r,t)$ is the vector of aggregate tax payments made by consumers to the $n-m$ revenue systems of economy $\mathcal{E}(t)$. Each component of ϕ corresponds to a separate revenue system. As the notation suggests consumer taxes may depend on which demand point is selected if demands are multi-valued. The breakdown of demand among individual consumers is immaterial, however (see Section 3.6 for an extension of the model which recognizes this distinction).

(e) A producer unit tax function $\gamma(\cdot,\cdot,t) : \mathcal{B}(t) \times S \to R^{n-m}$, homogeneous of degree one in (π,r). For each price-revenue pair (π,r) in S and each production activity b in $\mathcal{B}(t)$, $\gamma(b,\pi,r,t)$ is the vector of tax payments made by producers to the $n-m$ revenue systems of economy $\mathcal{E}(t)$ whenever activity b is operated at unit level. Each component of γ corresponds to a separate revenue system. Total producer taxes are determined by taking the same linear combinations of unit taxes that are taken to construct feasible production plans from unit activities. Thus the vector of aggregate taxes assessed against the production plan β defined in 3.1.1 is $\sum_{j=M+1}^{N} \gamma(b_j,\pi,r,t)\, y(j)$. Slack activities incur no tax liability.

The tax liability of a production plan may depend on the way the plan is expressed in terms of unit activities. For this reason each production plan β in pos $\alpha(t)$ must be accompanied by a particular representation whenever the plan appears in a tax context. The tax liability is unaffected, however, by the combination of producers which implement the plan. Hence the formulation does not cover the situation in which two producers with different tax rates operate the same unit activity. A straightforward extension of the model, however, can handle this case (see Section 3.6).

The dependence of ϕ and γ on (π,r,t) will frequently be suppressed in subsequent sections by abbreviating $\phi(\xi,\pi,r,t)$ and $\gamma(b,\pi,r,t)$ as $\phi(\xi)$ and $\gamma(b)$ respectively.

3.2. Definition of Equilibrium Graph

The concept of equilibrium for each economy $\mathcal{E}(t)$ is essentially the same as for a conventional competitive economy. Consumers maximize utility subject to a budget constraint, producers maximize after tax profits, and all markets and revenue systems clear. The concept of equilibrium graph for the family $\{\mathcal{E}(t)\}_{t\in T}$ is a natural extension of equilibrium for a single economy. An equilibirum graph consists simply of those points (π^*,r^*,t) in $S \times T$ for which (π^*,r^*) is an equilibrium price-revenue system for economy $\mathcal{E}(t)$. Supply and demand imputations accompanying equilibrium price-revenue pairs have been excluded from the definition of equilibrium graph for sake of conciseness.

3.2.1. DEFINITION. The triple $[(\pi^*, r^*), \xi^*, \beta^*]$ where $(\pi^*, r^*) \in S$; $\pi^* \neq 0$; $\xi^* \in \Xi(\pi^*, r^*, t)$; and $\beta^* = [-f_0^* \cdots -f_{M^*}^* \; b_{M^*+1}^* \cdots b_{N^*}^*]y^*$ where $N^* \geq 0$, $-1 \leq M^* \leq N^*$, $y^* \in R_+^{N^*+1}$, $f_j^* \in \mathcal{J}_{m+1}$ for $0 \leq j \leq M^*$, and $b_j^* \in \mathcal{B}(t)$ for $M^*+1 \leq j \leq N^*$ is said to be a competitive equilibrium for economy $\mathcal{E}(t)$ iff

(a) $\xi^* = \beta^* + w(t)$;

(b) $r^* = \sum_{j=M^*+1}^{N^*} \gamma(b_j^*, \pi^*, r^*, t)\, y^*(j) + \phi(\xi^*, \pi^*, r^*, t)$;

(c) $\pi^*\beta^* - \sum_{j=M^*+1}^{N^*} e\, \gamma(b_j^*, \pi^*, r^*, t)\, y^*(j) \geq \pi^*\beta - \sum_{j=M+1}^{N} e\, \gamma(b_j, \pi^*, r^*, t)\, y(j)$

for every β satisfying 3.1.1.

Relative equilibrium prices make little sense unless at least one price is positive. In the standard general equilibrium model relative prices lie on S and hence cannot all vanish. But here (π, r) lies on S so the condition $\pi^* \neq 0$ must be added. Condition (a) requires that supply equal demand in all commodity markets. Condition (b) requires that revenue disbursements equal gross tax receipts in each revenue system. Condition (c) requires that producers maximize after tax profits. Consumer utility maximization is implicit in a Walrasian demand correspondence. Walras Law for Ξ will be stated in Section 3.4.

3.2.2. DEFINITION. The set of all (π^*, r^*, t) in $S \times T$ such that (π^*, r^*) is an equilibrium price-revenue pair for $\mathcal{E}(t)$ constitutes the equilibrium graph of the family $\{\mathcal{E}(t)\}_{t \in T}$.

3.3. The Economic Algorithm

Now that the economic model has been specified, the next step is to invoke the fundamental algorithm of Chapter 2 to compute approximate equilibrium graphs. This will be accomplished by deriving from the economic model a labeling L of $S \times [0,\infty)$ and a vector p in R^{n+1} such that (L,p) form a proper pair. Before the labeling can be defined, however, a few technical restrictions must be placed on the economic constructs introduced in Section 3.1. These technical restrictions will be superceded by a set of economically meaningful restrictions in the next section.

Throughout the present section the index set T is assumed to be $[0,\infty)$, and an abstract pseudomanifold K^{n+1} on $S \times [0,\infty)$ is assumed to be given.

3.3.1. ASSUMPTION. The initial endowments of all economies are bounded above, i.e., $\exists\ W \in R^{m+1}$ s.t. $w(T) \leq W$.

3.3.2. ASSUMPTION. The demand correspondences of all economies are bounded below, i.e., $\exists\ d \in R^{m+1}$ s.t. $\Xi(S \times T) > d$.

3.3.3. ASSUMPTION. The combined production activities of the economies spanned by any $(n+1)$-simplex τ in K^{n+1} cannot generate any outputs unless inputs are provided, i.e., $\forall\ a_0, \ldots, a_{n+1}$ in $\mathcal{Q}(p_2(\tau))$, the linear inequality system $[a_0 \cdots a_{n+1}]y \geq 0$, $y \geq 0$ has only $y = 0$ as a solution.

3.3.4. REMARK. Since the disposal activities cannot be operated at positive levels without consuming resources, it suffices to verify the above condition for non-slack activities $a_0, \ldots, a_{n+1}$ in $\mathcal{B}(p_2(\tau))$.

3.3.5. REMARK. Since the linear inequality system in 3.3.3 is homogeneous, every similar linear inequality system $[a_0 \cdots a_{n+1}]y \geq b$, $y \geq 0$ is bounded for every b in R^{m+1}.

For each t in T let

$$c(t) = (1 + \|W\|_\infty)e + d^- - w(t) \tag{3.3.6}$$

and

$$\theta = c(t) + w(t) . \tag{3.3.7}$$

Then

$$\theta \gg 0 \tag{3.3.8}$$

and

$$\Xi(S,t) + c(t) > 0 . \tag{3.3.9}$$

The last inequality follows from the definition of $c(t)$ and from Assumptions 3.3.1 and 3.3.2.

Now choose Δ in R^{n-m} such that

$$\Delta \gg 0 \qquad \text{and} \qquad e\Delta < 1 . \tag{3.3.10}$$

Define p in R^{n+1} by

$$p = \begin{bmatrix} \theta \\ \Delta \end{bmatrix} \gg 0 . \tag{3.3.11}$$

The vector p will serve as the right hand side (RHS) of the linear inequality systems in the economic version of the fundamental algorithm.

Enough structure has now been imposed on the family $\{\mathcal{E}(t)\}_{t \in T}$ that an economic labeling L of $S \times [0,\infty)$ can be defined.

3.3.12. DEFINITION. Define $L : S \times [0,\infty) \to R^{n+1}$ by

$$L(\pi,r,t) = \begin{cases} e_j & \text{if } s = (\pi,r) \text{ lies on a facet of } S \\ & \text{and } j \text{ is the position of the } \underline{\text{last}} \\ & \text{zero in the } \underline{\text{first}} \text{ run of zeros in } s; \\ \begin{bmatrix} -b \\ \gamma(b) \end{bmatrix} & \text{if } (\pi,r) \gg 0 \text{ and } \pi b - e\gamma(b,\pi,r,t) > 0, \\ & \text{where } b \in \mathcal{B}(t). \\ \begin{bmatrix} \xi + c(t) \\ \phi(\xi) - r + \Delta \end{bmatrix} & \text{if } (\pi,r) \gg 0 \text{ and } \pi b - e\gamma(b,\pi,r,t) \leq 0 \\ & \text{for all } b \in \mathcal{B}(t), \text{ where } \xi \in \Xi(\pi,r,t). \end{cases}$$

In order that L be uniquely defined for each (π,r,t) in $S \times [0,\infty)$, a specific b and a specific ξ must be chosen in the second and third cases. In practice some tie-breaking procedure such as lexicographic minimization must be employed to prevent ambiguity in the evaluation of L.

Points in $S \times [0,\infty)$ which require the third case of the definition of L are called demand-labeled points. Those falling under the second case are called production-labeled points.

Occasionally in subsequent sections it will be convenient to partition the label $L(v)$ into

$$\begin{bmatrix} L_1(v) \\ L_2(v) \end{bmatrix}$$

where $L_1(v) \in R^{m+1}$ and $L_2(v) \in R^{n-m}$. Similarly the canonical vector e_j will be partitioned into $[{f \atop 0}]$ for $0 \leq j \leq m$ and $[{0 \atop g}]$ for $m+1 \leq j \leq n$.

The idea behind the L_1 portion of the labeling is due to Scarf. If some good is free then the label becomes the negative of the disposal activity for that good. If no goods are free and some activity earns a positive profit, then the label becomes the negative of that activity. If no goods are free and no activity earns a positive profit, then the label becomes a demand point. A technical difficulty arises when many goods are free, but this is overcome by the manner in which e_j is selected.

The L_2 portion of the labeling was originally conceived by Shoven and Whalley. The idea here is that revenue flows can be made to balance just like commodity flows by extending the production labels to include unit producer taxes and the demand labels to include consumer revenue receipts net of taxes.

The groundwork has now been laid for the main result of this section.

3.3.13. THEOREM. The pair (L,p) and the pseudomanifold K^{n+1} satisfy all the conditions required for the successful operation of the fundamental algorithm, namely (L,p) is a proper pair and Assumption 2.3.4 holds.

Proof: The vertices of $S \times \{0\}$ are $(e_0,0), \ldots, (e_n,0)$. The labels corresponding to these vertices are $e_n, e_0, e_1, \ldots, e_{n-1}$ respectively. Clearly the positive vector p lies in the cone spanned by these labels. Hence σ^0 is (L,p)-complete.

Now consider any facet of $S \times [0,\infty)$ other than $S \times \{0\}$, e.g., the facet F_j for $0 \leq j \leq n$. The only possible labels for points in F_j are the coordinate vectors $e_0, \ldots, e_n$. Consider a point (s,t) in F_j. If $1 \leq j \leq n$ then the $(j-1)$-st coordinate of s exists. Suppose $s(j-1) = 0$. Then $s(j-1)$ cannot be the last zero in a run of zeros since $s(j) = 0$. Suppose $s(j-1) > 0$. Then $s(j-1)$ does not appear in any run of zeros. In either case $L(s,t) \neq e_{j-1}$. If $j = 0$ then $s(n)$ cannot lie in the first run of zeros in s because $s \neq 0$. Hence $L(s,t) \neq e_n$. Whatever the value of j, one of the coordinate vectors cannot appear as a label for F_j. Since $p \gg 0$, the facet F_j is not (L,p)-complete.

All that remains is to verify Assumption 2.3.4. Let $\tau = \{v_0, \ldots, v_{n+1}\} \in K^{n+1}$. By suitably ordering τ the system $L(\tau)y = p$, $y \geq 0$ may be displayed as

$$(3.3.14)\quad \begin{bmatrix} 0 \cdots 0 & f_{H+1} \cdots f_I & -b_{I+1} \cdots -b_J & \xi_{J+1} + c(t_{J+1}) & \cdots & \xi_{n+1} + c(t_{n+1}) \\ g_0 \cdots g_H & 0 \cdots 0 & \gamma(b_{I+1}) \cdots \gamma(b_J) & \phi(\xi_{J+1}) - r_{J+1} + \Delta & \cdots & \phi(\xi_{n+1}) - r_{n+1} + \Delta \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} \theta \\ \Delta \end{bmatrix}$$

where $y_1 \in R_+^{H+1}$, $y_2 \in R_+^{I-H}$, $y_3 \in R_+^{J-I}$, $y_4 \in R_+^{n+1-J}$, and $-1 \le H \le I \le J \le n+1$. Any (y_1, y_2, y_3, y_4) satisfying 3.3.14 must also satisfy

$$(3.3.15)\quad [-f_{H+1} \cdots -f_I \;\; b_{I+1} \cdots b_J] \begin{bmatrix} y_2 \\ y_3 \end{bmatrix} \ge -\theta , \qquad y_2 \ge 0,\ y_3 \ge 0$$

because of 3.3.9. By Remark 3.3.5 the set of (y_2, y_3) satisfying 3.3.15 is bounded. Hence if y_4 satisfies 3.3.14, the expression $[\xi_{J+1} + c(t_{J+1}) \cdots \xi_{n+1} + c(t_{n+1})]y_4$ is bounded. By 3.3.9 each vector $\xi_j + c(t_j)$ for $J+1 \le j \le n+1$ is non-negative and contains a positive component, so y_4 must be bounded. The boundedness of (y_2, y_3, y_4) satisfying 3.3.14 implies the boundedness of $[g_0 \cdots g_H]y_1$, which in turn implies that y_1 is bounded. Therefore Assumption 2.3.4 holds. □

3.4. Limiting Behavior of the Algorithm

The theorem of the previous section guarantees the existence of an infinite sequence of distinct, adjacent, (L,p)-very complete n-simplices in $S \times [0,\infty)$. The connection between this sequence and the equilibrium graph of the family $\{\mathcal{E}(t)\}_{t\in T}$ remains to be demonstrated. Unless more structure is imposed on $\{\mathcal{E}(t)\}_{t\in T}$ and on the pseudomanifold K^{n+1}, the sequence may well be economically meaningless. Furthermore the model with $T = [0,\infty)$ is not the model of ultimate interest. Rather it serves as a tool for analyzing families of economies defined on $T = [0,1]$. These are the models toward which this study is primarily directed.

In this section enough restrictions will be placed on $\{\mathcal{E}(t)\}_{t\in [0,1]}$ to insure that the family possesses a non-void equilbrium graph. Then $\{\mathcal{E}(t)\}_{t\in [0,1]}$ will be copied onto the cylinder $S \times [0,\infty)$ in such a way that the conditions of the previous section are met. The sequence of n-simplices generated by the fundamental algorithm will then be mapped back to $S \times [0,1]$, where the images will be shown to cluster around a connected subset of the equilibrium graph of $\{\mathcal{E}(t)\}_{t\in [0,1]}$.

Throughout the remainder of this chapter the index set T is assumed to be $[0,1]$.

3.4.1. ASSUMPTION. Initial endowments $w(t)$ vary continuously in t.

This is the first of many assumptions stemming from the notion that $\{\mathcal{E}(t)\}_{t\in T}$ is generated by a continuous deformation.

3.4.2. ASSUMPTION. The market demand correspondence Ξ satisfies the following conditions:

(a) Ξ is u.s.c. on $S \times T$;

(b) $\Xi(v)$ is convex for all v in $S \times T$;

(c) $\Xi(\cdot, t)$ satisfies Walras Law for all t in T provided $\pi \gg 0$,

i.e., $\forall\ \xi \in \Xi(\pi, r, t)$, $\pi\xi + e\emptyset(\xi, \pi, r, t) = \pi w(t) + er$;

(d) Ξ is bounded from below on $S \times T$;

(e) As $(\pi, r, t) \to (\pi^*, r^*, t^*)$ in $S \times T$ with $(\pi, r) \gg 0$, if

$\limsup \mathrm{diam}(\Xi(\pi, r, t) \cup \{0\}) = \infty$, then $\lim \mathrm{dist}(\Xi(\pi, r, t), 0) = \infty$.

Upper semi-continuity of Ξ with respect to t is related to the deformation interpretation of the model. Parts (b) and (c) and upper semicontinuity w.r.t. (π, r) are standard properties of demand correspondences arising from utility maximization subject to a budget constraint. Assumption (d) is regularly employed in conventional general equilibrium models, e.g., [2] and [5]. Unlike conventional treatments of the model, however, no truncation arguments (or boundedness assumptions in the case of Scarf [13]) are required for the demand correspondences considered here. This advance is made possible by condition (e) which rules out pathological singularities of Ξ on the boundary of the price-revenue simplex. As long as condition (e) is satisfied, global feasiblity constraints will be automatically enforced by the algorithm.

3.4.3. ASSUMPTION. The non-slack production activity correspondence $\mathcal{B}$ satisfies the following conditions:

(a) $\mathcal{B}$ is continuous on T;

(b) $\mathcal{B}$ is a bounded correspondence.

Part (a) reflects the deformation interpretation of the model. Part (b) is a technical convenience and has no effect on production technology since unit activities may be operated at any non-negative level.

3.4.4. ASSUMPTION. For each t in T there exists $\epsilon_t > 0$ such that for all $a_0, \ldots, a_{n+1}$ in $\mathcal{a}(T \cap [t-\epsilon_t, t+\epsilon_t])$, the linear inequality system $[a_0 \cdots a_{n+1}]y \geq 0$, $y \geq 0$ has only $y = 0$ as a solution. This assumption captures the notions of continuity between economies and realism of production technology. It says that the combined production activities of economies sufficiently near a given economy cannot be operated at positive levels unless inputs are supplied.

3.4.5. REMARK. Analogues of Remarks 3.3.4 and 3.3.5 apply to Assumption 3.4.4.

3.4.6. ASSUMPTION. The consumer tax function ϕ satisfies the following conditions:

(a) $\phi \geq 0$;

(b) ϕ is continuous on $\bigcup_{(s,t) \in S \times T} \Xi(s,t) \times \{(s,t)\}$;

(c) ϕ vanishes when $\pi = 0$;

(d) ϕ is affine on $\Xi(s,t) \times \{(s,t)\}$ for fixed values of (s,t).

Part (a) rules out the possiblity of tax revenue flowing from revenue handling agents to consumers. Continuity w.r.t. t in part (b) reflects the continuity of change between economies. Part (c) relates the taxes paid by consumers to the value of their transactions. Part (d) and the remainder of part (b) are technical assumptions required for the main existence proof. Note that (d) becomes superfluous when demands are single valued. Despite these restrictions the function ϕ encompasses a wide class of possible tax schemes, including all those proposed in [15].

3.4.7. ASSUMPTION. The producer unit tax function γ satisfies the following conditions:

(a) $\gamma \geq 0$;

(b) γ is continuous on $\bigcup_{t \in T} \mathcal{B}(t) \times S \times \{t\}$.

(c) γ vanishes when $\pi = 0$.

Part (a) permits tax revenues to flow from producers to revenue handling agents but not vice-versa. This precludes the possibility of using γ to model direct producer subsidies but is necessary for technical reasons. The t component of part (b) again reflects the deformation aspects of $\{\mathcal{E}(t)\}_{t \in T}$. The remainder of part (b) is a technical assumption. Part (c) relates the level of producer taxes to the value of producer transactions.

Some elementary consequences of the preceding assumptions are contained in the following lemma.

3.4.8. LEMMA. The following sets are compact:

(a) $\mathcal{B}(t)$ for each t in T;

(b) $\mathcal{B}(T)$;

(c) $\mathcal{Q}(T \cap [t-\epsilon_t, t+\epsilon_t])$;

(d) $\bigcup_{t \in T} \mathcal{B}(t) \times S \times \{t\}$.

Proof: Since $-\mathcal{S}_{m+1}$ is compact, part (c) follows once $\mathcal{B}(T \cap [t-\epsilon_t, t+\epsilon_t])$ is shown to be compact. But this set is just $\bigcup_{u \in T \cap [t-\epsilon_t, t+\epsilon_t]} \mathcal{B}(u)$, which is compact by Lemma A.1. Likewise $\mathcal{B}(t) = \bigcup_{u \in T \cap \{t\}} \mathcal{B}(u)$, $\mathcal{B}(T) = \bigcup_{u \in T} \mathcal{B}(u)$, and the union in (d) are compact. □

Now construct a new family of economies $\{\mathcal{E}'(t')\}_{t'\in[0,\infty)}$ by letting $\mathcal{E}'(t') = \mathcal{E}(h(t'))$ where $h : [0,\infty) \to [0,1]$ is defined by

$$h(t) = \begin{cases} 0 \, , & 0 \leq t < 2I \\ t-i \, , & 2I \leq i \leq t < i+1 \text{ and } i \text{ even} \\ i+1-t \, , & 2I \leq i \leq t < i+1 \text{ and } i \text{ odd} \end{cases} \tag{3.4.9}$$

for some I in Z_+. Figure 3.4.1 illustrates how h copies the unprimed family of economies onto $S \times [0,\infty)$ to form the primed family.

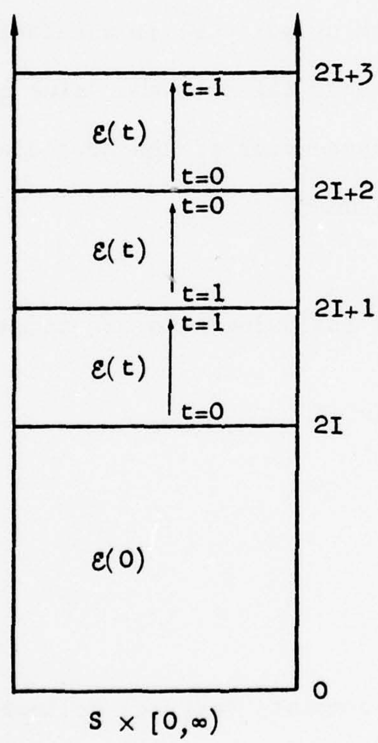


Figure 3.4.1. The family $\{\mathcal{E}'(t')\}_{t'\in[0,\infty)}$

In view of 3.4.1 and 3.4.2(d) the family $\{\mathcal{E}'(t')\}_{t'\in[0,\infty)}$ satifies Assumptions 3.3.1 and 3.3.2. Before Assumption 3.3.3 can be verified, an additional restriction is required on the pseudomanifold K^{n+1}.

3.4.10. ASSUMPTION. K^{n+1} becomes increasingly refined as $t \to \infty$, i.e., diam $\tau \to 0$ uniformly in $\tau \in K^{n+1}$ as $p_2(\tau) \to \infty$.

3.4.11 PROPOSITION. Provided the integer I in 3.4.9 is sufficiently large, the family $\{\mathcal{E}'(t')\}_{t'\in[0,\infty)}$ satisfies 3.3.3.

<u>Proof</u>: The collection $\{(t-\epsilon_t, t+\epsilon_t)\}_{t\in T}$ forms an open covering of T. Let δ be a Lebesgue number for this covering (recall that $T = [0,1]$). Choose I so large that for all τ in K^{n+1} satisfying $p_2(\tau) \cap [2I,\infty) \neq \emptyset$, diam $\tau < \delta$. Then for each τ in K^{n+1} there exists t in T such that $h(p_2(\tau)) \subset T \cap (t-\epsilon_t, t+\epsilon_t)$. In view of Assumption 3.4.4, Assumption 3.3.3 holds for $\{\mathcal{E}'(t')\}_{t'\in[0,\infty)}$. □

All the Assumptions of Section 3.3 have now been verified for $\{\mathcal{E}'(t')\}_{t'\in[0,\infty)}$. Hence the economic version of the fundamental algorithm developed in that section can be operated with the primed family to generate an infinite sequence $\langle\rho^k\rangle$ of distinct, adjacent, completely labeled n-simplices in $S \times [0,\infty)$.

In order to analyze the implications of the sequence $\langle\rho^k\rangle$ for the unprimed family of economies, it is necessary to map the sequence $\langle\rho^k\rangle$

back to $S \times T$ via the function $1_S \times h : S \times [0,\infty) \to S \times [0,1]$, where 1_S is the identity on S. Denote by σ^k the image of ρ^k under this mapping. Assign to σ^k the label system associated with ρ^k, and denote this label system as $L(\sigma^k)y^k = p,\ y^k \geq 0$.

The net effect of copying the unprimed family of economies onto $S \times [0,\infty)$ and then transforming back to $S \times T$ is the same as if the pseudomanifold K^{n+1} had first been mapped onto $S \times T$ by $1_S \times h$, and the development of Section 3.3 had taken place there. Hence expressions 3.3.6 through 3.3.11 apply for the unprimed family.

Before stating and proving the main result of this section, the uniform boundedness of the label systems $L(\sigma^k)y^k = p,\ y^k \geq 0$ will now be established. The proof resembles the boundedness argument in Theorem 3.3.13.

3.4.12. LEMMA. Let $t \in T$ and $S_t = \{\sigma \in \langle\sigma^k\rangle : p_2(\sigma) \subset T \cap [t-\epsilon_t, t+\epsilon_t]\}$. There exist bounded sets Y_t, $Z_t \subset R^{n+1}$ which contain, respectively, every solution y to every linear inequality system $L(\sigma)y = p,\ y \geq 0$ for σ in S_t, and all vectors

$$\begin{bmatrix} \xi_j \\ \phi(\xi_j) \end{bmatrix} y(j)$$

appearing in these systems.

<u>Proof</u>: For any σ in S_t the label system $L(\sigma)y = p,\ y \geq 0$ can be displayed as

$$(3.4.13)\quad \begin{bmatrix} 0 \cdots 0 & f_{H+1} \cdots f_I & -b_{I+1} \cdots -b_J & \xi_{J+1} + c(t_{J+1}) & \cdots \xi_n + c(t_n) \\ g_0 \cdots g_H & 0 \cdots 0 & \gamma(b_{I+1}) \cdots \gamma(b_J) & \phi(\xi_{J+1}) - r_{J+1} + \Delta & \cdots \phi(\xi_n) - r_n + \Delta \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} \theta \\ \Delta \end{bmatrix}$$

where $y_1 \in R_+^{H+1}$, $y_2 \in R_+^{I-H}$, $y_3 \in R_+^{J-I}$, $y_4 \in R_+^{n-J}$, and $-1 \leq H \leq I \leq J \leq n$. Because of 3.3.9 any (y_1, y_2, y_3, y_4) satisfying 3.4.13 must also satisfy

$$(3.4.14)\quad [-f_{H+1} \cdots -f_I \;\; b_{I+1} \cdots b_J] \begin{bmatrix} y_2 \\ y_3 \end{bmatrix} \geq -\theta\ , \quad y_2 \geq 0, \quad y_3 \geq 0\ .$$

By 3.3.8 and 3.4.5 any system of the form 3.4.14 with matrix columns taken from $\mathcal{A}(T \cap [t-\epsilon_t, t+\epsilon_t])$ is feasible and bounded. Therefore the hypotheses of Lemma A.3 are satisfied with $\mathcal{B} = \{-\theta\}$ and $C = \mathcal{A}(T \cap [t-\epsilon_t, t+\epsilon_t])$, which is compact by 3.4.8(c). Hence there exists a fixed bounded set containing all solutions to all systems of the form 3.4.14.

The compactness of $\mathcal{A}(T \cap [t-\epsilon_t, t+\epsilon_t])$ and the uniform boundedness of (y_2,y_3) imply that $[\xi_{J+1} + c(t_{J+1}) \cdots \xi_n + c(t_n)]y_4$ is uniformly bounded over all σ in S_t. (So, too, are each of the non-negative terms $(\xi_j + c(t_j))\, y(j)$ for $J+1 \leq j \leq n$.) Using a graph projection argument similar to the one in Lemma A.1, the set $\bigcup_{t \in T} \Xi(S,t) + c(t)$ is readily seen to be closed, and by 3.3.9 this set does not contain 0. Hence there exists a neighborhood of 0 with ℓ_∞-diameter $\epsilon > 0$ which misses the union. Any vector $\xi + c(t)$ in $\Xi(S,t) + c(t)$ for $t \in T$ must, therefore, contain a positive component of magnitude at least ϵ, and this places a uniform upper bound on the components of y_4 satisfying 3.4.13.

Since ϕ and γ are non-negative, the terms $\gamma(b_j)$ for $I+1 \leq j \leq J$ and $\phi(\xi_j) - r_j + \Delta$ for $J+1 \leq j \leq n$ are uniformly bounded below over all systems 3.4.13 for all σ in S_t. This fact together with the uniform boundedness of y_3 and y_4 implies that both $[g_0 \cdots g_H]y_1$ and $[\phi(\xi_{J+1}) - r_{J+1} + \Delta \cdots \phi(\xi_n) - r_n + \Delta]y_4$ are uniformly bounded over all systems 3.4.13. The uniform boundedness of y_1 follows from that of the first expression, and the uniform boundedness of $(\phi(\xi_j)-r_j+\Delta)y(j)$ for $J+1 \leq j \leq n$ from that of the second. The set Y_t may be taken as the cartesian product of the sets which uniformly bound y_1, y_2, y_3 and y_4. The existence of Z_t is assured by the uniform boundedness of $c(t_j)\, y(j)$ and $(\Delta-r_j)\, y(j)$ for $J+1 \leq j \leq n$, together with the uniform boundedness of the weighted labels containing these terms. $\square$

The following theorem contains the principal result of this section, namely the clustering of the sequence $\langle\sigma^k\rangle$ around a component of the equilibrium graph of $\{\mathcal{E}(t)\}_{t\in T}$.

3.4.15. THEOREM. The set Λ of limit points of $\langle\sigma^k\rangle$ in $S \times [0,1]$ is a connected subset of the equilibrium graph of $\{\mathcal{E}(t)\}_{t\in T}$ and it meets both $S \times \{0\}$ and $S \times \{1\}$. Furthermore for each (π^*, r^*, t) in Λ and each subsequence $\sigma^{k'} \to (\pi^*, r^*, t)$, an equilibrium consumption plan ξ^* in $\Xi(\pi^*, r^*, t)$ and an equilibrium production plan β^* in pos $\mathcal{Q}(t)$ may be obtained by taking linear combinations of limit points of the labels $L(\sigma^{k'})$, using weights which are limit points of the weights $y^{k'}$.

Proof: The proof is rather long and complicated, so it will be broken down into a series of nine steps.

Step 1: Λ is connected and meets both $S \times \{0\}$ and $S \times \{1\}$.

Since the sequence $\langle\rho^k\rangle \subset S \times [0,\infty)$ consists of adjacent n-simplices, each ρ^k contains one vertex missing from ρ^{k-1}. Let v^k denote the image of this vertex under the mapping $1_S \times h$. Then clearly the limit points of $\langle v^k\rangle$ coincide with the limit points of $\langle\sigma^k\rangle$. According to Corollary 2.4.2, $p_2(\rho^k) \to \infty$ as $k \to \infty$, and hence by Assumption 3.4.10, diam $\rho^k \to 0$ as $k \to \infty$. Therefore $\langle\rho^k\rangle$ eventually crosses each slice $S \times \{i\}$ for $i \in Z_+$, and as $i \to \infty$, dist$(\langle\rho^k\rangle, S \times \{i\}) \to 0$. Corresponding properties of the image sequence $\langle v^k\rangle$ are $\|v^{k+1} - v^k\| \to 0$, dist$(v^k, S \times \{0\}) \to 0$, and dist$(v^k, S \times \{1\}) \to 0$. Applying Lemma A.4 with $X = S \times T$, $A = S \times \{0\}$, $B = S \times \{1\}$, and $\langle x^k\rangle = \langle v^k\rangle$ yields the desired result.

Step 2: Extraction of convergent subsequences.

Let (π^*, r^*, t) in Λ be given. Select a subsequence of $\langle\sigma^k\rangle$, for convenience also indexed by k in Z_+, such that $\sigma^k \to (\pi^*, r^*, t)$. Since $\operatorname{diam} \sigma^k \to 0$, the subsequence may be chosen so that $p_2(\langle\sigma^k\rangle) \subset T \cap [t-\epsilon_t, t+\epsilon_t]$. Write σ^k as $\{v_0^k, \ldots, v_n^k\}$ and consider the label matrices $L(\sigma^k) = [L(v_0^k) \cdots L(v_n^k)]$. For $0 \le j \le n$ and k in Z_+ either

(i) $L_1(v_j^k) \in \{0\}$,

(ii) $L_1(v_j^k) \in \mathcal{J}_{m+1}$,

(iii) $L_1(v_j^k) \in -\mathcal{B}(T)$, or

(iv) $L_1(v_j^k) \in \Xi(S \times T) + c(T)$.

Hence for each $0 \le j \le n$, the vector $L_1(v_j^k)$ must lie in one of the above sets for infinitely many k in Z_+. In cases (i), (ii), and (iii), the containing sets are compact, so every infinite sequence of labels lying in one of them has a convergent subsequence within that set. Beginning with $j = 0$ and continuing until $j = n$, one may extract successive subsequences of $\langle\sigma^k\rangle$ until there remains a subsequence, for convenience also indexed by k in Z_+, such that for each $0 \le j \le n$ precisely one of the following four statements holds:

(a) $L_1(v_j^k) = 0$;

(b) $L_1(v_j^k) = f_j^k \equiv f_j^* \in \mathcal{J}_{m+1}$;

(c) $L_1(v_j^k) = -b_j^k \to -b_j^* \in -\mathcal{B}(t)$;

(d) $L_1(v_j^k) = \xi_j^k + c(t_j^k) \in \Xi(\pi_j^k, r_j^k, t_j^k) + c(t_j^k)$.

The inclusion relation in (c) follows from the u.s.c. of $\mathcal{B}$. Later it will be shown that the labels in (d) also contain convergent subsequences.

By suitably ordering the elements of each σ^k one may assume that

(a) holds for $0 \leq j \leq H$,

(b) holds for $H+1 \leq j \leq I$,

(c) holds for $I+1 \leq j \leq J$, and

(d) holds for $J+1 \leq j \leq n$,

where $-1 \leq H \leq I \leq J \leq n$. By extracting further subsequences of $\langle\sigma^k\rangle$, for convenience also indexed by k in Z_+, the following can be guaranteed to hold:

(a') $\quad L_2(v_j^k) = g_j^k \equiv g_j^* \in \mathcal{J}_{n-m} \qquad$ for $0 \leq j \leq H$.

Also, by definition of L_2, H, and I

(b') $\quad L_2(v_j^k) = 0 \qquad$ for $H+1 \leq j \leq I$.

By the continuity of γ,

(c') $\quad L_2(v_j^k) = \gamma(b_j^k) \rightarrow \gamma(b_j^*) \qquad$ for $I+1 \leq j \leq J$.

(The symbol $\gamma(b_j^*)$ is an abbreviation for $\gamma(b_j^*, \pi^*, r^*, t)$.)

Since $p_2(\langle\sigma^k\rangle) \subset T \cap [t-\epsilon_t, t+\epsilon_t]$, Lemma 3.4.12 insures that the solutions y^k to the linear inequality systems $L(\sigma^k)y^k = p$, $y \geq 0$ lie in a fixed bounded set, as do the terms $\xi_j^k\, y^k(j)$ and $\phi(\xi_j^k)\, y^k(j)$ for $J+1 \leq j \leq n$. Hence there exists a final subsequence of $\langle\sigma^k\rangle$, also indexed by k in Z_+, along which

(e) $\xi_j^k \, y^k(j) \to \bar{\xi}_j^* \in R^{m+1}$ for $J+1 \leq j \leq n$;

(e') $\phi(\xi_j^k) \, y^k(j) \to \bar{\phi}_j^* \in R^{n-m}$ for $J+1 \leq j \leq n$;

(f) $y^k \to y^* = \begin{bmatrix} y_0^* \\ y_1^* \end{bmatrix} \in R_+^{n+1}$, where $y_0^* \in R_+^{J+1}$ and $y_1^* \in R_+^{n-J}$.

The end result of this extraction procedure is that along the final subsequence $\langle \sigma^k \rangle$, the label systems $L(\sigma^k)y^k = p$, $y^k \geq 0$ converge componentwise to the system

$$(3.4.16) \quad \begin{bmatrix} 0 \cdots 0 & f_{H+1}^* \cdots f_I^* & -b_{I+1}^* \cdots b_J^* \\ g_0^* \cdots g_H^* & 0 \cdots 0 & \gamma(b_{I+1}^*) \cdots \gamma(b_J^*) \end{bmatrix} y_0^*$$

$$+ \sum_{j=J+1}^{n} \left\{ \begin{bmatrix} \bar{\xi}_j^* \\ \bar{\phi}_j^* \end{bmatrix} + \begin{bmatrix} c(t) \\ \Delta - r^* \end{bmatrix} y^*(j) \right\} = \begin{bmatrix} \theta \\ \Delta \end{bmatrix}, \; y^* \geq 0 .$$

The remainder of the proof consists of showing that the triple $[(\pi^*, r^*), \xi^*, \beta^*]$, where $\xi^* = \sum_{j=J+1}^{n} \bar{\xi}_j^*$ and $\beta^* = - \sum_{j=H+1}^{I} f_j^* \, y^*(j) + \sum_{j=I+1}^{J} b_j^* \, y^*(j)$, constitutes a competitive equilibrium for economy $\mathcal{E}(t)$.

<u>Step 3</u>: $\pi^* f_j^* = 0$ for $H+1 \leq j \leq I$;

$$\pi^* b_j^* - e\gamma(b_j^*) \geq 0 \qquad \text{for } I+1 \leq j \leq J .$$

For each k in Z_+ and $H+1 \leq j \leq I$, $\pi_j^k f_j^k = 0$. Since $\pi_j^k \to \pi^*$ and $f_j^k = f_j^*$, it follows that $\pi^* f_j^* = 0$. For each k in Z_+ and $I+1 \leq j \leq J$, the definition of L implies $\pi_j^k b_j^k - e\gamma(b_j^k) > 0$. Letting $k \to \infty$ and invoking the continuity of γ, this inequality becomes $\pi^* b_j^* - e\gamma(b_j^*) \geq 0$.

Step 4: $J < n$.

Suppose otherwise. In view of the previous step and Assumption 3.4.7(a)

$$\pi^* b_j^* \geq e\gamma(b_j^*) \geq 0 \qquad \text{for} \quad I+1 \leq j \leq J .$$

Multiplying 3.4.16 by $(\pi^*, 0)$ and applying the first part of Step 3 yields

$$- \sum_{j=I+1}^{J} \pi^* b_j^* y^*(j) = \pi^* \theta ,$$

which is a contradiction since the RHS is positive and the LHS (left hand side) is non-positive.

Step 5: After-tax profits in economy $\mathcal{E}(t)$ are maximized at prices and revenue levels (π^*, r^*) by the production plan

$$\beta^* = - \sum_{j=H+1}^{I} f_j^* y^*(j) + \sum_{j=I+1}^{J} b_j^* y^*(j) ,$$

and these profits are exactly zero.

Since $J < n$, there exists j' in $\{J+1, \ldots, n\}$ such that $\forall$ $k \in Z_+$,

$$L_1(v_{j'}^k) = \xi_{j'}^k + c(t_{j'}^k) .$$

Let b in $\mathcal{B}(t)$ be given. By the l.s.c. of $\mathcal{B}$ there exists a sequence $\langle b^k_{j'} \rangle$ such that $b^k_{j'} \in \mathcal{B}(t^k_{j'})$ and $b^k_{j'} \to b$ as $k \to \infty$. By definition of L,

$$\pi^k_{j'} b^k_{j'} - e\gamma(b^k_{j'}) \leq 0 ,$$

so letting $k \to \infty$ yields

$$\pi^* b - e\gamma(b) \leq 0 . \tag{3.4.17}$$

Now consider the arbitrary production plan β defined in 3.1.1. At prices and revenue levels (π^*, r^*) the after-tax profitability of β is

$$- \sum_{j=0}^{M} \pi^* f_j \, y(j) + \sum_{j=M+1}^{N} [\pi^* b_j - e\gamma(b_j)] \, y(j) ,$$

which is non-positive in light of 3.4.17.

The results of Step 3 together with 3.4.17 imply that the after-tax profitability of each unit activity in β^* is zero. Therefore the profitability of β^* exceeds that of any other production plan β. Condition (c) in the definition of a competitive equilibrium has now been verified.

<u>Step 6:</u> $\sum_{j=0}^{H} y^*(j) = 0;\quad \sum_{j=J+1}^{n} y^*(j) = 1.$

Multiplying system 3.4.16 by (π^*, e) and applying the zero profit results of the previous step yields

$$(3.4.18)\qquad \sum_{j=0}^{H} y^*(j) + \sum_{j=J+1}^{n} [(\pi^* c(t) + e\Delta - er^*)\, y^*(j) + \pi^* \bar{\xi}_j^* + e\bar{\phi}_j^*]$$
$$= \pi^*\theta + e\Delta .$$

For $J+1 \leq j \leq n$ the definition of L insures that $\pi_j^k >> 0$, hence Walras Law [3.4.2(c)] implies

$$\pi_j^k\, \xi_j^k\, y^k(j) + e\phi(\xi_j^k)\, y^k(j) = \pi_j^k\, w(t_j^k)\, y^k(j) + er_j^k\, y^k(j) \quad .$$

Letting $k \to \infty$ this equation becomes

$$\pi^*\, \bar{\xi}_j^* + e\bar{\phi}_j^* = \pi^*\, w(t)\, y^*(j) + er^*\, y^*(j) \; .$$

Solving for $\pi^* w(t)\, y^*(j)$ and replacing the last three terms in brackets in 3.4.18 yields

$$(3.4.19)\qquad \sum_{j=0}^{H} y^*(j) + (\pi^*\theta + e\Delta) \sum_{j=J+1}^{n} y^*(j) = \pi^*\theta + e\Delta \; .$$

Now let $q = \operatorname{sgn} r^*$. For sufficiently large k, $q \leq \operatorname{sgn} r_j^k$, and hence $qg_j^* = 0$ for $0 \leq j \leq H$. Multiplying 3.4.16 by $(0, e-q)$ and subtracting the resulting equation from 3.4.19 yields

$$- \sum_{j=I+1}^{J} (e-q)\, \gamma(b_j^*)\, y^*(j) + \sum_{j=J+1}^{n} [(\pi^*\theta + q\Delta)\, y^*(j) - (e-q)\, \bar{\phi}_j^*]$$

$$= \pi^*\theta + q\Delta \,.$$

By the non-negativity of γ and ϕ this implies

$$(\pi^*\theta + q\Delta) \sum_{j=J+1}^{n} y^*(j) \geq \pi^*\theta + q\Delta \,, \quad \text{or} \quad \sum_{j=J+1}^{n} y^*(j) \geq 1 \;.$$

Combining this inequality with 3.4.19 implies $\sum_{j=0}^{H} y^*(j) = 0$. Dividing 3.4.19 by $\pi^*\theta + e\Delta$ then establishes the desired result.

Step 7: The sequences $\langle \xi_j^k \rangle$ for $J+1 \leq j \leq n$ contain convergent subsequences.

Suppose $\langle \xi_{j'}^k \rangle$ is unbounded for some $j' \in \{J+1, \ldots, n\}$. Then there exists a subsequence along which $\|\xi_{j'}^k\| \to \infty$. According to Assumption 3.4.2(e), the other demand points ξ_j^k for $J+1 \leq j \leq n$ must also diverge to $+\infty$ along this subsequence. Since $\xi_j^k\, y^k(j) \to \bar{\xi}_j^*$, it follows that for $J+1 \leq j \leq n$, $y^k(j) \to 0$ along the subsequence, contradicting the fact that $\sum_{j=J+1}^{n} y^*(j) = 1$.

Consequently every sequence $\langle \xi_j^k \rangle$ for $J+1 \leq j \leq n$ is bounded, and thus contains a limit point ξ_j^*. By extracting a further subsequence of $\langle \sigma^k \rangle$, also indexed by $k \in Z_+$, the sequences $\langle \xi_j^k \rangle$ may be considered to converge to their limit points, i.e.,

$$\xi_j^k \to \xi_j^* \in \Xi\,(\pi^*, r^*, t) \qquad \text{for } J+1 \leq j \leq n \;.$$

The inclusion relation follows from the u.s.c. of Ξ. Since ϕ is continuous on its domain, it also follows that $\phi(\xi_j^k) \to \phi(\xi_j^*, \pi^*, r^*, t)$, henceforth abbreviated $\phi(\xi_j^*)$. The weighted label limits $\bar{\xi}_j^*$ and $\bar{\phi}_j^*$ may now be factored into $\bar{\xi}_j^* = \xi_j^* \, y^*(j)$ and $\bar{\phi}_j^* = \phi(\xi_j^*) \, y^*(j)$ for $J+1 \leq j \leq n$.

Step 8: $\pi^* \not\leq 0$.

Suppose $\pi^* = 0$. Then $qr^* = 1$ (recall the properties of q from Step 6). Also, Assumptions 3.4.6(c) and 3.4.7(c) require that $\phi(\xi_j^*) = 0$ for $J+1 \leq j \leq n$ and $\gamma(b_j^*) = 0$ for $I+1 \leq j \leq J$. Multiplying system 3.4.16 on the left by $(0,q)$ yields

$$(-1 + q\Delta) \sum_{j=J+1}^{n} y^*(j) = q\Delta \; ,$$

which is impossible since $y^* \geq 0$ and $0 < q\Delta < 1$.

Step 9: Verification of equilibrium conditions 3.2.1(a) and 3.2.1(b).

Since $\sum_{j=J+1}^{n} y^*(j) = 1$ and $\Xi(\pi^*, r^*, t)$ is convex, the demand point

$$\xi^* = \sum_{j=J+1}^{n} \bar{\xi}_j^* = \sum_{j=J+1}^{n} \xi_j^* \, y^*(j)$$

belongs to $\Xi(\pi^*, r^*, t)$. The first $m+1$ equations of system 3.4.16 reduce to

$$\sum_{j=H+1}^{I} f_j^* y^*(j) - \sum_{j=I+1}^{J} b_j^* y^*(j) + \xi^* + c(t) = w(t) + c(t) .$$

Subtracting $c(t)$ from both sides yields equilibrium condition 3.2.1(a).

Assumption 3.4.6(d) implies that

$$\sum_{j=J+1}^{n} \phi(\xi_j^*) y^*(j) = \phi\left(\sum_{j=J+1}^{n} \xi_j^* y^*(j)\right) = \phi(\xi^*) .$$

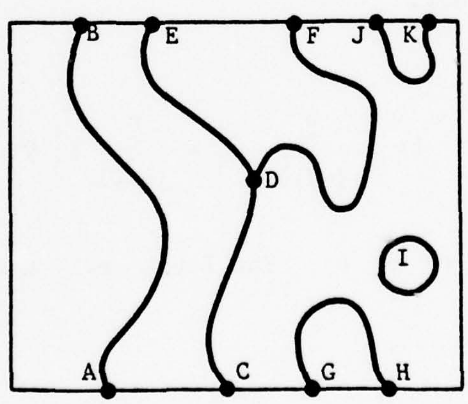
Thus the last $n-m$ equations of system 3.4.16 become

$$\sum_{j=I+1}^{J} \gamma(b_j^*) y^*(j) + \phi(\xi^*) - r^* + \Delta = \Delta .$$

Subtracting Δ from both sides yields equilibrium condition 3.2.1(b).

The proof of Theorem 3.4.15 is now complete. □

Figure 3.4.2 illustrates the relationship between the set Λ and the equilibrium graph as a whole. The graph in the figure consists of arcs AB, CDE, DF, GH, JK and loop I. The set Λ may coincide with the arc AB, with the arc CDE plus any subarc of DF emanating from D, or



$S_1 \times [0,1]$

Figure 3.4.2. A complicated equilibrium graph.

with the arc CDF plus any subarc of DE emanating from D. One would expect equilibrium graphs encountered in practice to be much simpler than the one illustrated here and hence more nearly coincident with Λ.

3.5. Finite Approximation of Equilibrium Graph

The purpose of this section is to show that an economically meaningful approximation to the equilibrium graph of $\{\mathcal{E}(t)\}_{t \in T}$ can be constructed from a finite segment of $\langle \sigma^k \rangle$ whenever the diameters of the sets comprising the segment are sufficiently small. Such approximations are essential if the algorithm is to have any practical value, since cluster points can rarely be computed. The role of the theorem in the previous section is to insure that the approximations proposed below relate to something which actually exists. Two additional restrictions on the market demand correspondence Ξ are needed to guarantee certain desirable properties of the approximations. These restrictions lead to regularity conditions analogous to uniform continuity for Ξ and the consumer tax function ϕ.

Recall from Section 3.4 the sequence $\langle \rho^k \rangle$ of n-simplices in $S \times [0,\infty)$ generated by the economic algorithm. Consider any block $S \times [i, i+1]$ where $2I \leq i$. Since $p_2(\rho^k) \to \infty$, there exists a last simplex $\rho^{\underline{k}}$ which meets $S \times [0,i]$ and a first simplex $\rho^{\bar{k}}$ following $\rho^{\underline{k}}$ which meets $S \times [i+1, \infty)$.

3.5.1. **DEFINITION.** The set $\Lambda_i = \bigcup_{k=\underline{k}}^{\bar{k}} \mathcal{D}(\sigma^k)$, where $\mathcal{D}(\sigma^k)$ is the set of demand labeled vertices of σ^k, is defined to be the i-th level approximation to the equilibrium graph of $\{\mathcal{E}(t)\}_{t \in T}$.

Two criteria must be satisfied before the finite set Λ_i can be considered a reasonable approximate equilibrium graph. First Λ_i must cover the continuum of economies, and second the price and revenue components of points in Λ_i must induce economic behavior resembling equilibrium. The economies may be considered well-covered when the index of every economy lies close to a point in $p_2(\Lambda_i)$. Economic behavior resembles equilibrium when supply is close to demand and unit profits are nearly maximized. The theorem below shows that for simplices σ^k of sufficiently small diameter, the sets $\mathcal{D}(\sigma^k)$ are non-empty, and the price-revenue levels of points therein induce approximate equilibrium behavior to any desired accuracy. Thus by choosing i sufficiently large (so that the diameters of the sets σ^k are sufficiently small), the set Λ_i can be made to approximate a component of the equilibrium graph of $\{\mathcal{E}(t)\}_{t \in T}$ arbitrarily well. As always, however, with equation solving techniques of this type, the location of the true equilibrium graph can never be determined precisely unless extra regularity conditions are imposed on the model.

The additional restrictions on Ξ required for the proof of the approximation theorem are rather technical and non-intuitive. They are motivated, however, by the consumption example discussed in Section 5.1. Their main purpose is to induce a type of uniform continuity on Ξ. This is particularly difficult since Ξ may very well be unbounded (over a compact domain). It is, therefore, necessary to restrict Ξ to subsets of $S \times T$ where its size can be controlled.

To this end let $\mathcal{L}(\alpha) = \{v \in S \times T : \mathrm{dist}_\infty(\Xi(v), 0) \leq \alpha\}$. This set is simply the projection onto $S \times T$ of the compact set formed by intersecting the graph of Ξ with the closed ℓ_∞-ball of diameter α, and is hence compact.

3.5.2. ASSUMPTION. For each $\alpha \geq 0$ there exists a closed subset $\underline{\mathcal{L}}(\alpha)$ of $\mathcal{L}(\alpha)$ satisfying

(a) $\underline{\mathcal{L}}(\alpha)$ contains all points (π, r, t) in $\mathcal{L}(\alpha)$ with $\pi \gg 0$;

(b) Ξ is bounded on $\underline{\mathcal{L}}(\alpha)$;

(c) Ξ is l.s.c. on $\underline{\mathcal{L}}(\alpha)$.

3.5.3. ASSUMPTION. $\mathrm{Diam}_\infty\, \Xi(\pi, r, t)$ is bounded over $\{(\pi, r, t) \in S \times T : \pi \gg 0\}$.

The latter assumption is reminiscent of condition 3.4.2(e), and in fact when combined with 3.5.2 implies that condition (see Remark 3.5.16). All other conditions and constructions of Section 3.4 are assumed to remain in effect here.

3.5.4. THEOREM. Let $\lambda, \mu > 0$. Then $\exists\ \delta > 0$ such that any n-simplex σ in $\langle \sigma^k \rangle$ and its associated label system 3.4.13 exhibit the following properties whenever $\mathrm{diam}_1\ \sigma < \delta$.

(a) Demand-labeled vertices exist, i.e., $J < n$ in 3.4.13.

Let $(\pi, r, t) \in \sigma$ be such a vertex.

(b) There exists an actual production plan $\tilde{\beta}$ in pos $\mathcal{Q}(t)$ that is within ℓ_∞-distance μ of the pseudo-production plan

$$\beta = [-f_{H+1} \cdots -f_I \;\; b_{I+1} \cdots b_J] \begin{bmatrix} y_2 \\ y_3 \end{bmatrix}$$

constructed from 3.4.13, and such that at prices and revenue levels (π, r), the after-tax profitability of any unit activity in $\mathcal{Q}(t)$ exceeds the after-tax profitability of any unit activity used in $\tilde{\beta}$ by at most λ.

(c) There exists an actual demand point $\tilde{\xi}$ in $\Xi(\pi, r, t)$ that is within ℓ_∞-distance μ of the pseudo-demand point

$$\xi = \left(\sum_{j=J+1}^{n} y(j) \right)^{-1} \sum_{j=J+1}^{n} \xi_j \, y(j)$$

constructed from 3.4.13, and within ℓ_∞-distance 2μ of the aggregate supply $\tilde{\beta} + w(t)$.

(d) Actual tax receipts genezated by $\tilde{\beta}$ and $\tilde{\xi}$ are within ℓ_∞-distance μ of pseudo-tax receipts

$$\sum_{j=I+1}^{J} \gamma(b_j) \, y(j) + \sum_{j=J+1}^{n} \phi(\xi_j) \, y(j)$$

derived from 3.4.13, and within ℓ_∞-distance 2μ of actual revenue levels r.

Proof: As in Theorem 3.4.15 the proof is broken down into a series of steps.

Step 1: Selection of uniformity constants.

The family $\{(t-\epsilon_t, t+\epsilon_t)\}_{t\in T}$ forms an open covering of T. Let $\{(t_j-\epsilon_{t_j}, t_j+\epsilon_{t_j})\}_{j\in \mathcal{J}}$ be a finite subcover of T, and let δ_0 be a Lebesgue number for this subcover. By Lemma 3.4.12 there exist bounded sets Y_{t_j} and Z_{t_j} for each j in $\mathcal{J}$ which contain, respectively, all solutions y to all systems 3.4.13 formed by choosing σ in S_{t_j}, and all vectors

$$\begin{bmatrix} \xi_\ell \\ \phi(\xi_\ell) \end{bmatrix} y(\ell)$$

for $J+1 \le \ell \le n$ appearing in these systems. Let

(i) $C_1 > 0$ be an ℓ_∞-bound on $\mathcal{Q}(T)$;

(ii) $C_2 \ge 1$ be an ℓ_1-bound on $\bigcup_{j\in\mathcal{J}} Y_{t_j}$;

(iii) $C_3 > 0$ be an ℓ_∞-bound on $\bigcup_{j\in\mathcal{J}} Z_{t_j}$;

(iv) $C_4 \ge 1$ be an ℓ_∞-bound on $c(T)$;

(v) $C_5 > 0$ be the bound on $\operatorname{diam}_\infty \Xi$ postulated in 3.5.3;

(vi) $C_6 > (n+1)C_3 + C_5 + 1$;

(vii) $C_7 > 0$ be an ℓ_∞-bound on ϕ over $\bigcup_{v\in\underline{\mathcal{L}}(C_6)} \Xi(v) \times \{v\}$. ($C_7$ exists because of 3.5.2, Lemma A.1, and the continuity of ϕ.)

Choose $\epsilon > 0$ to satisfy

(viii) $\epsilon < \frac{\min_i p(i)}{2}\left(\frac{1}{(n+1)C_3} - \frac{1}{C_6 - C_5 - 1}\right)$;

(ix) $\frac{4\epsilon(\max_i p(i) + \epsilon\Delta + C_4 + C_5 + C_6 + C_7)}{\min_i p(i)} < \mu$.

Choose $\epsilon_1 > 0$ such that

(x) $\epsilon_1 < \lambda/2$;

(xi) $\epsilon_1 < \epsilon/C_2$.

Since γ is continuous on the compact set $\bigcup_{t \in T} \mathcal{B}(t) \times S \times \{t\}$, there exists a 1-∞ uniformity constant δ_1 for (γ, ϵ_1). (See A.6 for the definition of a p-q uniformity constant.) Choose $\epsilon_2 > 0$ to satisfy

(xii) $\epsilon_2 < \delta_1/2$;

(xiii) $\epsilon_2 < \mu/C_2$;

(xiv) $\epsilon_2 < \lambda/2 - \epsilon_1$;

(xv) $\epsilon_2 < \epsilon/C_2 - \epsilon_1$.

Since $\mathcal{B}$ is a continuous bounded correspondence, Lemma A.5 guarantees the existence of a 1-1 uniformity constant δ_2 for $(\mathcal{B}, \epsilon_2)$. Let $\epsilon_3 > 0$ be chosen so that

(xvi) $\epsilon_3 < \epsilon/C_2 - \epsilon_1$.

Since ϕ is continuous on the compact set $\bigcup_{v \in \underline{\mathcal{L}}(C_6)} \Xi(v) \times \{v\}$, there exists a 1-∞ uniformity constant δ_3 for (ϕ, ϵ_3). Choose $\epsilon_4 > 0$ to satisfy

(xvii) $\epsilon_4 < \mu$;

(xviii) $\epsilon_4 < \delta_3/2$;

(xix) $\epsilon_4 < \epsilon/C_2$.

Let $\delta_4 > 0$ be a 1-1 uniformity constant for (Ξ, ϵ_4) on $\mathfrak{L}(C_6)$. Choose $\epsilon_5 > 0$ so that

(xx) $\epsilon_5 < \epsilon/C_2 - \epsilon_4$

and let $\delta_5 > 0$ be a 1-∞ uniformity constant for (c, ϵ_5).

Let $\delta_6 > 0$ be a 1-∞ uniformity constant for $(\Xi, 1)$ on $\mathfrak{L}(C_6)$. Finally choose $\delta > 0$ to satisfy

(xxi) $\delta < \delta_0$;

(xxii) $\delta < \delta_1/2$;

(xxiii) $\delta < \delta_2$;

(xxiv) $\delta < \delta_3/2$;

(xxv) $\delta < \delta_4/2$;

(xxvi) $\delta < \delta_5$;

(xxvii) $\delta < \delta_6$;

(xxviii) $\delta < \dfrac{\min_i p(i)}{2C_1C_2}$;

(xxix) $\delta < \dfrac{1-e\Delta}{2(n-m+1)}$;

(xxx) $\delta < \dfrac{1}{C_1}(\lambda/2 - \epsilon_1 - \epsilon_2)$;

(xxxi) $\delta < \dfrac{\epsilon/C_2 - \epsilon_1 - \epsilon_2}{C_1 + C_3 + C_4 + \|\theta\|_\infty + 2(n-m)}$;

(xxxii) $\delta < \dfrac{1}{C_2}\left(\mu - \dfrac{4\epsilon(\max_i p(i) + e\Delta + 1)}{\min_i p(i)}\right)$.

Consider any σ in $\langle\sigma^k\rangle$ with $\text{diam}_1\ \sigma < \delta$. Recall that $\sigma = ((\pi_0, r_0, t_0), \ldots, (\pi_n, r_n, t_n))$. Let (π, r, t) be any point in $\text{conv}\ \sigma$. Put $q = \text{sgn}(r-\delta e)^+$.

<u>Step 2</u>: $qg_j = 0$ for $0 \leq j \leq H$ and $\pi\theta + q\Delta > \frac{1}{2}\min_i p(i)$.

Since $\|r-r_j\|_1 < \delta$ for $0 \leq j \leq n$, whenever $q(i) = 1$, i.e., $r(i) > \delta$, then $r_j(i) > \delta-\delta = 0$. The first assertion follows from the definition of L.

If $q = 0$, i.e., $r \leq \delta e$, then

$$\begin{aligned} \|\pi\|_1 &= 1 - \|r\|_1 \\ &\geq 1 - (n-m)\delta \\ &> \frac{1}{2}, \qquad \text{by (xxix)} . \end{aligned}$$

Consequently,

$$\begin{aligned} \pi\theta + q\Delta &\geq \|\pi\|_1 \min_i p(i) \\ &> \frac{1}{2} \min_i p(i) . \end{aligned}$$

Since the inequality holds trivially when $q > 0$, the second assertion is also established.

<u>Step 3</u>: Construction of $\tilde{\beta}$.

In view of (xxiii) there exist activities $\tilde{b}_j$ in $\mathcal{B}(t)$ for $I+1 \leq j \leq J$ such that $\|\tilde{b}_j - b_j\|_1 < \epsilon_2$. The production plan

$$\tilde{\beta} = [-f_{H+1} \cdots -f_I \;\; \tilde{b}_{I+1} \cdots \tilde{b}_J] \begin{bmatrix} y_2 \\ y_3 \end{bmatrix}$$

belongs to pos $\mathcal{Q}(t)$ and satisfies

$$(3.5.5) \quad \|\tilde{\beta}-\beta\|_\infty \leq \|\tilde{\beta}-\beta\|_1$$

$$= \Big\|\Big(- \sum_{j=H+1}^{I} f_j \, y(j) + \sum_{j=I+1}^{J} \tilde{b}_j \, y(j)\Big)$$

$$- \Big(- \sum_{j=H+1}^{I} f_j \, y(j) + \sum_{j=I+1}^{J} b_j \, y(j)\Big)\Big\|_1$$

$$\leq \sum_{j=I+1}^{J} \|\tilde{b}_j - b_j\|_1 \, y(j) \leq \epsilon_2 \, C_2 < \mu \ .$$

The C_2 factor arises because of (xxi), and the last inequality follows from (xiii).

<u>Step 4</u>: Lower bounds on unit profitabilities.

For $I+1 \leq j \leq J$,

$$|\pi b_j - \pi_j b_j| \leq \|\pi - \pi_j\|_1 \, \|b_j\|_\infty \leq \delta C_1 \ .$$

Since $\pi_j b_j - e\gamma(b_j) > 0$, $\pi b_j - e\gamma(b_j) \geq -\delta C_1$. For $H+1 \leq j \leq I$,

$$|\pi f_j - \pi_j f_j| \leq \|\pi - \pi_j\|_1 \, \|f_j\|_\infty \leq \delta C_1 \ .$$

Since $\pi_j f_j = 0$, $-\pi f_j \geq -\delta C_1$.

<u>Step 5</u>: $\sum_{j=J+1}^{n} y(j) \not\leq 0.$

Suppose otherwise. Recall from Step 2 that $qg_j = 0$ for $0 \leq j \leq H.$ Multiplying 3.4.13 by (π, q) yields

$$0 = \pi\theta + q\Delta - \sum_{j=H+1}^{I} \pi f_j \, y(j) + \sum_{j=I+1}^{J} [\pi b_j - q\gamma(b_j)] \, y(j)$$

$$\geq \pi\theta + q\Delta - \delta C_1 \sum_{j=H+1}^{J} y(j)$$

$$\geq \frac{1}{2} \min_i p(i) - \delta C_1 C_2 > 0 \ .$$

The first inequality follows from the previous step, the second from Step 2, and the third from (xxviii). The claim follows by contradiction. Part (a) of the theorem is now established.

<u>Step 6</u>: Upper bounds on unit profitabilities.

In view of the preceding step, there exists j' in $\{J+1, \ldots, n\}$ such that

$$L(v_{j'}) = \xi_{j'} + c(t_{j'}) \in \Xi(v_{j'}) + c(t_{j'}) \ .$$

Let b be an arbitrary unit activity in $\mathcal{B}(t)$. Because of (xxiii) there exists $b_{j'}$ in $\mathcal{B}(t_{j'})$ s.t. $\|b-b_{j'}\|_1 < \epsilon_2$. Hence

$$|[\pi b - e\gamma(b,v)] - [\pi_{j'} b_{j'} - e\gamma(b_{j'}, v_{j'})]|$$

$$= |[\pi b - \pi b_{j'}] + [\pi b_{j'} - \pi_{j'} b_{j'}] + e[\gamma(b_{j'}, v_{j'}) - \gamma(b,v)]|$$

$$\leq \|\pi\|_1 \|b-b_{j'}\|_\infty + \|\pi-\pi_{j'}\|_1 \|b_{j'}\|_\infty + \epsilon_1$$

$$\leq \epsilon_2 + \delta C_1 + \epsilon_1 \quad .$$

The ϵ_1 term is a consequence of (xii) and (xxii). The ϵ_2 term appears because $\|\cdot\|_\infty \leq \|\cdot\|_1$. Since $\pi_{j'} b_{j'} - e\gamma(b_{j'}, v_{j'}) \leq 0$, it follows that

$$\pi b - e\gamma(b,v) \leq \delta C_1 + \epsilon_1 + \epsilon_2 \quad . \tag{3.5.6}$$

For $I+1 \leq j \leq J$,

$$|[\pi \tilde{b}_j - e\gamma(\tilde{b}_j, v)] - [\pi_j b_j - e\gamma(b_j, v_j)]|$$

$$= |[\pi \tilde{b}_j - \pi_j \tilde{b}_j] + [\pi_j \tilde{b}_j - \pi_j b_j] + e[\gamma(b_j, v_j) - \gamma(\tilde{b}_j, v)]|$$

$$\leq \|\pi-\pi_j\|_1 \|\tilde{b}_j\|_\infty + \|\pi_j\|_1 \|\tilde{b}_j - b_j\|_\infty + \epsilon_1$$

$$\leq \delta C_1 + \epsilon_2 + \epsilon_1 \quad .$$

Since $\pi_j b_j - e\gamma(b_j, v_j) > 0$, it follows that

$$-(\delta C_1 + \epsilon_1 + \epsilon_2) \leq \pi \tilde{b}_j - e\gamma(\tilde{b}_j, v) \quad . \tag{3.5.7}$$

Adding inequalities 3.5.6 and 3.5.7 yields

$$(3.5.8) \qquad \pi b - e\gamma(b,v) \leq \pi\tilde{b}_j - e\gamma(\tilde{b}_j,v) + 2(\delta C_1 + \epsilon_1 + \epsilon_2)$$

$$\leq \pi\tilde{b}_j - e\gamma(\tilde{b}_j, v) + \lambda .$$

The last step uses inequality (xxx). By a straightforward extension of the preceding argument, inequality 3.5.8 also holds if the RHS is replaced by $-\pi f_j + \lambda$ for any $H+1 \leq j \leq I$, or the LHS is replaced by $-\pi f$ for any f in $\mathcal{J}_{m+1}$, or both. Hence $\tilde{\beta}$ exhibits the desired unit profitability property, and part (b) of the theorem is established.

<u>Step 7</u>: $\left|1 - \sum_{j=J+1}^{n} y(j)\right| \leq \frac{\epsilon}{\pi\theta + q\Delta}$ and $\sum_{j=0}^{H} y(j) \leq \frac{\epsilon(2\pi\theta + (e+q)\Delta)}{\pi\theta + q\Delta}$.

Consider the expression

$$(3.5.9) \qquad \Bigg|(\pi\theta + e\Delta) \sum_{j=J+1}^{n} y(j) + \sum_{j=0}^{H} y(j) + \sum_{j=H+1}^{I} \pi f_j\, y(j)$$

$$- \sum_{j=I+1}^{J} [\pi b_j - e\gamma(b_j)]y(j) - (\pi\theta + e\Delta)\Bigg| .$$

Multiplying 3.4.13 by (π, e) and solving for the last four terms of 3.5.9, then substituting this value back into 3.5.9 and cancelling the $e\Delta$ terms yields

$$(3.5.10) \qquad \left|\pi\theta \sum_{j=J+1}^{n} y(j) - \sum_{j=J+1}^{n} [\pi\xi_j + \pi c(t_j) + e\phi(\xi_j) - er_j]\, y(j)\right| .$$

Since $\pi_j \gg 0$ for $J+1 \leq j \leq n$, Walras Law [3.4.2(c)] implies

$$\sum_{j=J+1}^{n} [\pi_j \xi_j + \pi_j\, c(t_j) + e\phi(\xi_j) - er_j]\; y(j) = \sum_{j=J+1}^{n} \pi_j \theta\, y(j) \;.$$

Adding the LHS and subtracting the RHS of this equation from expression 3.5.10 yields

$$\left| \sum_{j=J+1}^{n} (\pi - \pi_j)\; \theta y(j) - \sum_{j=J+1}^{n} (\pi - \pi_j)\; [\xi_j + c(t_j)]\; y(j) \right| ,$$

which is bounded by

$$\delta \|\theta\|_\infty C_2 + \delta C_3 + \delta C_4 C_2 \;,$$

and this in turn (since $C_2 \geq 1$) by

$$\delta C_2 (C_3 + C_4 + \|\theta\|_\infty) \;. \tag{3.5.11}$$

The bounds are direct applications of (ii), (iii), and (iv).

One consequence of the domination of 3.5.9 by 3.5.11 is that

$$\begin{aligned}(\pi\theta + e\Delta) \sum_{j=J+1}^{n} y(j) + \sum_{j=0}^{H} y(j) &\geq - \sum_{j=H+1}^{I} \pi f_j\; y(j) + \sum_{j=I+1}^{J} [\pi b_j - e\gamma(b_j)]\; y(j) \\ &\qquad + (\pi\theta + e\Delta) - \delta C_2 (C_3 + C_4 + \|\theta\|_\infty) \\ &\geq (\pi\theta + e\Delta) - \delta C_1 C_2 - \delta C_2 (C_3 + C_4 + \|\theta\|_\infty) \;.\end{aligned} \tag{3.5.12}$$

The second inequality follows from Step 4.

A second consequence of the domination of 3.5.9 by 3.5.11 is that

$$(3.5.13) \qquad (\pi\theta + e\Delta) \sum_{j=J+1}^{n} y(j) + \sum_{j=0}^{H} y(j)$$

$$\leq - \sum_{j=H+1}^{I} \pi f_j \, y(j) + \sum_{j=I+1}^{J} [\pi b_j - e\gamma(b_j)] \, y(j)$$

$$+ (\pi\theta + e\Delta) + \delta C_2(C_3 + C_4 + \|\theta\|_\infty)$$

$$\leq (\pi\theta + e\Delta) + (\delta C_1 + \epsilon_1 + \epsilon_2)C_2 + \delta C_2(C_3 + C_4 + \|\theta\|_\infty)$$

$$\leq (\pi\theta + e\Delta) + \epsilon \, .$$

The second inequality follows from an obvious extension of 3.5.6 to vectors b in $\mathcal{B}(t_j)$ for $I+1 \leq j \leq J$, and the third inequality from (xxxi).

Multiplying 3.4.13 by (0, e-q), solving for $\sum_{j=0}^{H} y(j)$, and substituting this expression into 3.5.12 yields

$$(\pi\theta + e\Delta) \sum_{j=J+1}^{n} y(j) \geq \sum_{j=I+1}^{J} (e-q) \, \gamma(b_j) \, y(j)$$

$$+ \sum_{j=J+1}^{n} [(e-q) \, \phi(\xi_j) - (e-q)r_j + (e-q)\Delta] \, y(j)$$

$$- (e-q)\Delta + (\pi\theta + e\Delta) - \delta C_2(C_1 + C_3 + C_4 + \|\theta\|_\infty) \, .$$

Since γ and ϕ are non-negative and $(e-q)r_j \leq 2(n-m)\delta$, the preceding inequality implies

$$(3.5.14) \qquad (\pi\theta + q\Delta) \sum_{j=J+1}^{n} y(j) \geq (\pi\theta + q\Delta) - 2(n-m)\,\delta C_2 - \delta C_2(C_1+C_3+C_4 + \|\theta\|_\infty)$$

$$\geq (\pi\theta + q\Delta) - \epsilon \ , \qquad \text{by (xxxi)} \ .$$

Dropping the non-negative term $\sum_{j=0}^{H} y(j)$ from 3.5.13, dividing by $\pi\theta + e\Delta$, and combining the result with $(\pi\theta + q\Delta)^{-1}$ times 3.5.14 yields

$$\frac{\pi\theta + q\Delta - \epsilon}{\pi\theta + q\Delta} \leq \sum_{j=J+1}^{n} y(j) \leq \frac{\pi\theta + e\Delta + \epsilon}{\pi\theta + e\Delta} \ ,$$

which implies

$$\frac{-\epsilon}{\pi\theta + e\Delta} \leq 1 - \sum_{j=J+1}^{n} y(j) \leq \frac{\epsilon}{\pi\theta + q\Delta} \ .$$

Replacing the left hand term by the more negative term $\frac{-\epsilon}{\pi\theta + q\Delta}$ completes the first half of this step.

Rearranging 3.5.13 and imposing the lower bound on $\sum_{j=J+1}^{n} y(j)$ derived from 3.5.14 yields

$$\sum_{j=0}^{H} y(j) \leq \pi\theta + e\Delta + \epsilon - (\pi\theta + e\Delta)\,\frac{\pi\theta + q\Delta - \epsilon}{\pi\theta + q\Delta}$$

$$= \frac{\epsilon(2\pi\theta + (e+q)\Delta)}{\pi\theta + q\Delta} \ ,$$

thereby completing the second half of this step.

Step 8: $\|\xi_j\|_\infty \leq C_6$ for $J+1 \leq j \leq n$.

Suppose there exists $j' \in \{J+1, \ldots, n\}$ such that $\|\xi_{j'}\|_\infty > C_6$. Let v_j be any of the other demand labeled vertices of σ. Then either $\|\xi_j\|_\infty > C_6$ or else $v_j \in \mathcal{L}(C_6)$. In the latter case since $\pi_j \gg 0$, the vertex v_j actually belongs to $\underline{\mathcal{L}}(C_6)$ by 3.5.2(a).

Consider the line segment joining $v_{j'}$ and v_j. All points (π,r,t) on this line segment have $\pi \gg 0$, and hence are subject to Walras Law [3.4.2(c)]. The right hand terms of the Walrasian equation are bounded on the segment; the left hand terms are bounded below. Hence the left hand terms are also bounded, in particular the term $\pi\Xi(\pi,r,t)$. Since the line segment (ignoring t) lies in the interior of S, it follows that Ξ is bounded there. Thus the entire segment lies in $\underline{\mathcal{L}}(\alpha)$ for some $\alpha > 0$, and by 3.5.2(c) the demand correspondence is l.s.c. on the line segment.

Since $\underline{\mathcal{L}}(C_6)$ is closed, it meets the line segment in a closed set. Hence there is a point $\bar{v}$ in the intersection closest to the end point $v_{j'}$. If $\bar{v} \neq v_{j'}$, then no point in $\Xi(\bar{v})$ can be less than C_6 in ℓ_∞-norm. Otherwise, since Ξ is l.s.c. there, a point on the segment slightly closer to $v_{j'}$ than $\bar{v}$ can be found whose image under Ξ also contains a point with ℓ_∞-norm less than C_6, thus contradicting the definition of $\bar{v}$. Hence if $\bar{v} \neq v_{j'}$, there exists a point $\bar{\xi}$ in $\Xi(\bar{v})$ with ℓ_∞-norm at least C_6. The same is trivially true if $\bar{v} = v_{j'}$, e.g., let $\bar{\xi} = \xi_{j'}$.

Since $\|\bar{v}-v_j\|_1 < \delta < \delta_6$, there is a point $\bar{\xi}_j$ in $\Xi(v_j)$ within ℓ_∞-distance 1 of $\bar{\xi}$. Hence $\|\bar{\xi}_j\|_\infty \geq C_6 - 1$, and by the definition of C_5, $\|\xi_j\|_\infty \geq C_6 - C_5 - 1$.

It has now been established that $\|\xi_{j'}\|_\infty > C_6$ implies that for all $J+1 \leq j \leq n$, either $\|\xi_j\|_\infty > C_6$ or else $\|\xi_j\|_\infty \geq C_6 - C_5 - 1$. By definition of C_3,

$$C_3 \geq \|\xi_j\, y(j)\|_\infty = \|\xi_j\|_\infty\, y(j) \geq (C_6 - C_5 - 1)\, y(j) .$$

Hence $y(j) \leq C_3/(C_6 - C_5 - 1)$ for $J+1 \leq j \leq n$. But in light of Steps 2 and 7,

$$\begin{aligned} 1 - \frac{2\epsilon}{\min_i p(i)} &< 1 - \frac{\epsilon}{\pi\theta + q\Delta} \\ &\leq \sum_{j=J+1}^{n} y(j) \\ &\leq \frac{(n+1)\, C_3}{C_6 - C_5 - 1} , \end{aligned}$$

which contradicts (viii) and thereby completes the step.

Step 9: Construction of $\tilde{\xi}$.

The arbitrary point $(\pi, r, t) \in \operatorname{conv} \sigma$ chosen at the end of Step 1 will henceforth be considered one of the demand-labeled vertices (π_j, r_j, t_j) for $J+1 \leq j \leq n$. In view of the preceding step, all such points belong to $\mathfrak{L}(C_6)$. Clearly $\pi \neq 0$.

As a consequence of (xxv) there exist vectors $\tilde{\xi}_j$ in $\Xi(\pi, r, t)$ for $J+1 \leq j \leq n$ which satisfy $\|\tilde{\xi}_j - \xi_j\|_1 < \epsilon_4$. Define

$$\tilde{\xi} = \left(\sum_{j=J+1}^{n} y(j) \right)^{-1} \sum_{j=J+1}^{n} \tilde{\xi}_j \, y(j) \quad .$$

Since $\Xi(\pi,r,t)$ is convex, it contains $\tilde{\xi}$. Abbreviate $\phi(\tilde{\xi}_j,\pi,r,t)$ and $\phi(\tilde{\xi},\pi,r,t)$ by $\phi(\tilde{\xi}_j)$ and $\phi(\tilde{\xi})$ respectively.

Step 10: Supply-demand proximity.

Consider the expression

$$\text{(3.5.15)} \qquad \left\| \tilde{\xi} - w(t) + \sum_{j=H+1}^{I} f_j \, y(j) - \sum_{j=I+1}^{J} b_j \, y(j) \right\|_\infty \quad .$$

Upon adding and subtracting identical terms this becomes

$$\left\| \sum_{j=J+1}^{n} [\tilde{\xi} + c(t)] \, y(j) + \left(1 - \sum_{j=J+1}^{n} y(j)\right) (\tilde{\xi} + c(t)) - \theta \right.$$
$$\left. + \sum_{j=H+1}^{I} f_j \, y(j) - \sum_{j=I+1}^{J} b_j \, y(j) \right\|_\infty \quad .$$

Expanding $\tilde{\xi}$ according to its definition and replacing the last three terms by equivalent terms taken from 3.4.13 yields

$$\left\| \sum_{j=J+1}^{n} [\tilde{\xi}_j - \xi_j + c(t) - c(t_j)] \, y(j) + \left(1 - \sum_{j=J+1}^{n} y(j)\right) (\tilde{\xi} + c(t)) \right\|_\infty ,$$

which in view of the first part of Step 7 is bounded by

$$(\epsilon_4 + \epsilon_5)C_2 + \frac{\epsilon(C_4 + C_5 + C_6)}{\pi\theta + q\Delta}$$

$$\leq \frac{\epsilon(\pi\theta + q\Delta + C_4 + C_5 + C_6)}{\pi\theta + q\Delta}$$

$$\leq \frac{2\epsilon(\max_i p(i) + e\Delta + C_4 + C_5 + C_6)}{\min_i p(i)} < \mu \quad .$$

The second inequality uses (xx), the third uses Step 2, and the last uses (ix). Combining inequality 3.5.5 with the one just established, i.e., μ exceeds expression 3.5.15, yields

$$\|\tilde{\xi} - w(t) - \tilde{\beta}\|_\infty < 2\mu \quad ,$$

thereby proving the second half of part (c). The first half is established by the following calculation:

$$\left\|\tilde{\xi} - \left(\sum_{j=J+1}^{n} y(j)\right)^{-1} \sum_{j=J+1}^{n} \xi_j \, y(j)\right\|_\infty$$

$$= \left(\sum_{j=J+1}^{n} y(j)\right)^{-1} \left\|\sum_{j=J+1}^{n} [\tilde{\xi}_j - \xi_j] \, y(j)\right\|_\infty$$

$$\leq \left(\sum_{j=J+1}^{n} y(j)\right)^{-1} \epsilon_4 \sum_{j=J+1}^{n} y(j) = \epsilon_4 < \mu \; .$$

The final inequality comes from (xvii).

Step 11: Tax-revenue proximity.

The discrepancy between the actual taxes levied against $\tilde{\xi}$ and $\tilde{\beta}$, and the pseudo-taxes defined in part (d) may be estimated as follows:

$$\left\| (\phi(\tilde{\xi}) + \sum_{j=I+1}^{J} \gamma(\tilde{b}_j)\, y(j)) - (\sum_{j=J+1}^{n} \phi(\xi_j)\, y(j) + \sum_{j=I+1}^{J} \gamma(b_j)\, y(j)) \right\|_\infty$$

$$= \left\| \sum_{j=J+1}^{n} [\phi(\tilde{\xi}_j) - \phi(\xi_j)]\, y(j) + (1 - \sum_{j=J+1}^{n} y(j))\, \phi(\tilde{\xi}) + \sum_{j=I+1}^{J} [\gamma(\tilde{b}_j) - \gamma(b_j)]\, y(j) \right\|_\infty$$

$$\leq \epsilon_3 C_2 + \frac{\epsilon C_7}{\pi\theta + q\Delta} + \epsilon_1 C_2$$

$$\leq \frac{\epsilon(\pi\theta + q\Delta + C_7)}{\pi\theta + q\Delta} \leq \frac{2\epsilon(\max_i p(i) + e\Delta + C_7)}{\min_i p(i)} < \mu \quad .$$

The equality step uses the affineness of ϕ in the demand coordinate. The first inequality uses the first part of Step 7 plus (xviii), (xxiv), (xxii), and the definitions of $\tilde{\xi}_j$ for $J+1 \leq j \leq n$ and $\tilde{b}_j$ for $I+1 \leq j \leq J$. The second inequality uses (xvi), and the third inequality Step 2. The final inequality follows from (ix).

The discrepancy between actual revenue and pseudo-taxes may be estimated by

$$\left\| r - \left(\sum_{j=J+1}^{n} \phi(\xi_j)\, y(j) + \sum_{j=I+1}^{J} \gamma(b_j)\, y(j) \right) \right\|_\infty$$

$$= \left\| r + \sum_{j=0}^{H} g_j\, y(j) + \sum_{j=J+1}^{n} [\Delta - r_j]\, y(j) - \Delta \right\|_\infty$$

$$= \left\| \sum_{j=J+1}^{n} [r - r_j]\, y(j) + \left(1 - \sum_{j=J+1}^{n} y(j)\right)(r - \Delta) + \sum_{j=0}^{H} g_j\, y(j) \right\|_\infty$$

$$\leq \delta C_2 + \frac{\epsilon}{\pi\theta + q\Delta} + \frac{\epsilon(2\pi\theta + (e+q)\Delta)}{\pi\theta + q\Delta}$$

$$\leq \delta C_2 + \frac{\epsilon(2\pi\theta + 2e\Delta + 2)}{\pi\theta + q\Delta}$$

$$\leq \delta C_2 + \frac{4\epsilon(\max_i p(i) + e\Delta + 1)}{\min_i p(i)} < \mu \quad .$$

The first equality is obtained by a substitution involving the last $n-m$ equations of 3.4.13. The second equality is a result of algebraic manipulation. The first inequality uses all of Step 7 plus the fact that r and Δ are bounded above by e. The last two inequalities follow from Step 2 and (xxxii) respectively.

Combining the two estimates obtained above yields

$$\left\| r - \left(\phi(\tilde{\xi}) + \sum_{j=I+1}^{J} \gamma(\tilde{b}_j)\, y(j) \right) \right\|_\infty < 2\mu \quad .$$

Part (d) of the theorem is now established, thereby completing the proof. □

3.5.16. REMARK. The argument in Step 8 additionally shows that Ξ satisfies condition 3.4.2(e).

3.5.17. REMARK. If Ξ happens to be bounded on $S \times T$, then Assumptions 3.5.2 and 3.5.3 can be replaced by the single assumption of lower semi-continuity. Furthermore, the conclusions hold for any (π, r, t) in conv σ (not just demand-labeled vertices), and $\|\pi\|_1 \geq \delta$.

The above theorem guarantees that the economic algorithm will generate an approximate equilibrium graph of arbitrarily high quality in a finite number of steps. For any desired tolerances λ and μ, one need only operate the algorithm until it passes through a block $S \times [i, i+1]$ containing n-simplices of diameter no larger than δ. The last segment of $\langle \rho^k \rangle$ which spans the block would then be mapped back to $S \times T$ to yield an approximate equilibrium graph Λ_i. Since neighboring points in Λ_i differ by at most 2δ in ℓ_1-norm, every economy $\mathcal{E}(t)$ lies within δ of an economy $\mathcal{E}(t_j^k)$ represented by a point (π_j^k, r_j^k, t_j^k) in Λ_i. The latter economy has production and consumption plans which comprise an approximate equilibrium (modulo λ and μ) at prices and revenue levels (π, r), and which may be approximated (within μ) by pieces of the label system $L(\sigma^k)y^k = p$, $y^k \geq 0$.

Of course in particular numerical examples the uniformity constant δ is never known. Even if it were, one would be foolish to operate the algorithm until a block $S \times [i, i+1]$ containing n-simplices of diameter no greater than δ was traversed. Instead one would continually monitor supply-demand imbalances and unit profit negativity in hopes of satisfying the tolerances μ and λ well before the diameters of the n-simplices reached δ. Without the knowledge that δ exists, however, the use of such criteria would be difficult to justify.

Figures 1.1.5 and 1.1.6 indicate what an approximate equilibrium graph might look like. The endpoints of the line segments forming the polygonal path comprise the sets σ^k. The approximate graph Λ_i consists of selected endpoints, including at least one from each σ^k.

3.6. Alternate Versions of the Model

The economic model presented in this chapter incorporates one or more revenue collection and distribution systems of the type introduced by Shoven and Whalley. The development of the model relies heavily on the presence of these systems. If, however, one wishes to study economies with no revenue systems, his best recourse within the context of the model is to let $n = m+1$ and set the tax functions $\emptyset$ and γ identically equal to zero. This will indeed induce the algorithm to correctly approximate the equilibrium graph of the family of economies. Such an approach is wasteful computationally, however, since the algorithm must operate in a space containing one unnecessary dimension.

A better approach is to strip the model of revenue systems entirely. The analysis in this chapter can easily be repeated for such a reduced model. All coordinates of S are allocated to prices, and the L_2 portion of the economic labeling disappears. The second equilibrium condition is discarded, and taxes are removed from the third. Walras Law is purged of the $\emptyset$ and r terms. All constructions and proofs are simpler and can be readily obtained by condensing existing constructions and proofs. Several of the numerical examples discussed in Chapter 6 are based on this variation of the economic model.

Another potentially useful variation recognizes differences in the tax situations of individual consumers and producers. As the model is presently formulated, taxes depend only on the aggregate behavior of economic agents. This simplification is satisfactory so long as each demand point and production plan result from unique combinations of agent behavior. But if two producers, for example, with different tax rates operate the same non-slack activity, then γ must necessarily be ill-defined.

This limitation can be easily overcome by assigning each consumer his own demand correspondence and tax function and each producer his own unit activity correspondence and unit tax function. These agent-specific mappings must obey the same rules as the present economy-wide mappings. Combining the agent-specific mappings by summing demands and taxes and taking unions of activity sets results in new economy-wide mappings which satisfy the same conditions as the present ones. Therefore the proofs go through virtually unchanged.

CHAPTER 4

COMPUTATIONAL REFINEMENTS

The development of the economic algorithm in the preceding chapter was directed primarily toward exploiting the labeling capabilities of the fundamental algorithm of Chapter 2. Little attention was paid to the abstract pseudomanifold K^{n+1} other than to ascribe it a needed refinement property. Before computations can be performed with the algorithm, however, an explicit specification of the pseudomanifold is required. Although refining subdivisions of $S \times [0,\infty)$ have been discovered which satisfy conditions 2.2.1 and 3.4.10, unfortunately none of these are practical for computing equilibrium graphs. The problem with conventional structures is that they refine rapidly and inexorably as $t \to \infty$. Consequently, the approximation error along equilibrium graphs produced on such structures tends to shrink to unmanageable levels before the final economy is reached.

The desire to maintain relatively uniform levels of approximation error along equilibrium graphs necessitates a new concept in manifold design. The dynamic definition principle introduced in the next section is such a concept. This principle essentially states that the geometry of the manifold need not be fixed in advance, but can be dynamically altered in response to accuracy requirements as the algorithm proceeds.

In Section 4.2 two families of manifolds which embody the dynamic definition principle are introduced -- one for $R^n \times [0,\infty)$ and another for $S \times [0,\infty)$. Each member of these families is constructed from transformed sections of Todd's J_1 and J_3 triangulations [20]. The configuration of these sections is determined by external requests based on information contained in the label systems $L(\sigma)y = p$, $y \geq 0$ for $\sigma \in K^n$. Section 4.3 describes how these label systems are manipulated to generate the manifold requests. The final section of the chapter lays out the basic architecture of the computer routines used to implement the economic algorithm, with special emphasis on the sequence of major processing activities and information flows between them.

4.1. The Uniform Approximation Problem

The quality of the finite approximations proposed in Section 3.5 is measured in terms of supply-demand and tax-revenue proximity, and unit profit negativity. A convenient term for describing these assorted deviations from equilibrium behavior is range error. The companion term, domain error, will be used to denote the diameters of the n-simplices σ^k from which the approximate equilibria are extracted. Phrased in this language, Theorem 3.5.4 states that all components of range error along an approximate equilibrium graph Λ_i can be made to satisfy pre-determined bounds by keeping the domain error sufficiently small. The theorem says nothing, however, about fluctuations of range error within the prescribed bounds.

For computational purposes it is highly desirable to have uniform levels of range error throughout the set Λ_i. The principal benefit of uniformity is computational efficiency, i.e., fewer iterations required to generate Λ_i. The numerical experiments reported in Chapter 6 reveal that large amounts of computational effort are required to produce equilibrium graphs for even modest size problems. Since manifold mesh is a key determinant of computational effort, it is imperative to use the largest grids possible which still keep range error within prescribed bounds. Also, holding range error relatively constant increases the likelihood that observed price variations along the equilibrium graph result from changes in economic behavior rather than from variations in the quality of approximations.

A fundamental computational issue is, therefore, how to maintain uniform levels of range error along the approximate equilibrium graph. The only readily controllable parameter which influences range error is domain error, i.e., the mesh of the manifold. There is no way, however, to determine in advance what the domain error should be along a particular path. What is needed is a dynamic adjustment mechanism to monitor range error as the algorithm proceeds and alter the manifold accordingly. Such a mechanism would have to rely on an external information loop to provide control data, because the range error of an approximate equilibrium depends on the entire label system $L(\sigma^k)y = p,\ y \geq 0$ evaluated at current prices, not just the most recent label.

The idea of dynamically adjusting the manifold presents some serious technical difficulties. First of all the manifold generated through such a procedure must satisfy conditions 2.2.1(a) - (d) in order for the algorithm to be assured of working— a non-trivial requirement even for rigidly defined structures. Furthermore, the way the manifold is defined

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in a particular region must be recorded so that the same definition can be used if the algorithm passes through that region again. In order to prevent the information structures which store the manifold definition from becoming unmanageable, some limitations must be placed on the type of adjustments that are permitted. If too little flexibility is allowed, however, then the ability to control range error is lost.

A compromise between complete flexibility and rigid pre-definition is achieved in the dynamically defined manifolds D_1 and D_2 introduced in the next section. Adjustments to the manifold mesh are permitted only when the algorithm moves upward in $S \times [0,\infty)$ to previously unattained levels. Whenever such a movement occurs, the manifold is immediately specified on a thin slab $S \times [t^k, t^{k+1}]$ protruding into virgin territory. The manifold may thus be regarded as a flexibly defined stack of rigidly defined layers. The price paid for this compromised freedom of definition is the potential loss of control over range error if the path $\langle \sigma^k \rangle$ turns back down into previously defined regions of the manifold. It is by no means inevitable, however, that such loss of control will be accompanied by a loss of accuracy.

The concept of flexible manifold definition may be summarized in the following dynamic definition principle: the specification of any portion of the manifold may be deferred until that portion is actually required for calculations, provided the specification procedure always results in a legitimate structure. This principle may be extended to the family of economies $\{\mathcal{E}(t)\}_{t \in T}$ as well. Hence one could conceivably build dynamic growth models in which the parameters of future economies depend on currently attained equilibria. Such models, however, are beyond the scope of this study.

An additional benefit of dynamic manifold construction is a simplification of the procedure outlined in Section 3.5 for locating a satisfactory approximate equilibrium graph. Instead of generating successive sets Λ_i until one satisfying all range error tolerances is found, only the single set Λ_{2I} need be computed. This is accomplished by refining the manifold until all range tolerances for economy $\mathcal{E}(0)$ are satisfied, then adjusting the manifold to maintain these tolerances as the algorithm proceeds through the remainder of the family $\{\mathcal{E}(t)\}_{t\in T}$. The level in $S \times [0,\infty)$ where the first acceptable equilibrium for $\mathcal{E}(0)$ appears is defined to be $2I$. The rate of progression through the remainder of the family is controlled by applying a vertical scale factor to the economy index.

4.2. Dynamically Defined Manifolds D_1 and D_2

The dynamic manifold construction procedure outlined above will now be formally laid out and analyzed. The analysis is based on a characterization of all possible outcomes of the procedure. Each outcome will be shown to constitute a legitimate pseudo-manifold on $S \times [0,\infty)$. Most of the development deals with triangulations instead of pseudomanifolds. The two concepts, however, are equivalent for all practical purposes.

The exposition begins by defining a family D_1 of triangulations of $R^n \times [0,\infty)$ which simultaneously triangulate a subset of $R^n \times [0,\infty)$

affinely homeomorphic to $S \times [0,\infty)$. The image of D_1 under the homeomorphism yields a family D_2 of triangulations of $S \times [0,\infty)$. Members of the class D_2 represent all possible outcomes of the dynamic construction process.

The raw materials from which the family D_1 is build are Todd's uniform triangulation J_1 and his refining triangulation J_3. Slabs of simplices from J_1 and J_3 are stacked together to exhaust $R^n \times [0,\infty)$. Care is taken so that the faces of simplices in adjacent slabs agree on the interface between the slabs. Much of the notation is taken from Todd's paper [20]. One minor change, however, is that the simplices considered here are closed, but have relatively open faces.

Throughout this section $R^n \times [0,\infty)$ will be identified with $\{x \in R^{n+1} : x(0) \geq 0\}$ and $S \times [0,\infty)$ with $\{x \in R_+^{n+2} : x(1) + \cdots + x(n+1) = 1\}$. The $[0,\infty)$ factor thus lies along the 0-th coordinate axis in both spaces. This factor will frequently be described as extending in the "vertical" direction.

The specification of D_1 begins with the primary building blocks $J_1(\delta)$ and J_3 and their derivatives. The triangulation $J_1(\delta)$ of R^{n+1} depending on the scale factor $\delta > 0$ is defined as follows. Let $J_1^0(\delta) = \{y \in R^{n+1} : y(i)/\delta$ is integral for $0 \leq i \leq n\}$ be the set of vertices of $J_1(\delta)$, and let $J_1^{0c}(\delta) = \{y \in J_1^0(\delta) : y(i)/\delta$ is odd for $0 \leq i \leq n\}$ be the set of central vertices.

4.2.1. DEFINITION. $J_1(\delta)$ consists of all $(n+1)$-simplices $\tau = (y_{-1}, y_0, \ldots, y_n)$ such that for some triple (y,ψ,a) in $J_1^{0c} \times \Psi_{n+1} \times A^{n+1}$ (see Section 1.3 for definitions of the latter two symbols)

$$y_{-1} = y$$

$$y_i = y_{i-1} + \delta a(i)\, e_{\psi(i)}, \qquad \text{for } 0 \leq i \leq n .$$

The simplex τ is also written $\tau = (y,\psi,a)$. In Lemma 3.2 of [20], Todd proves that $J_1(\delta)$ triangulates R^{n+1}.

An important hereditary property of $J_1(\delta)$ which insures that the slabs comprising members of D_1 fit together properly is noted in the following proposition.

4.2.2. PROPOSITION. The faces of simplices in $J_1(\delta)$ which meet the slice $R^n \times \{\delta\ell\}$ for ℓ odd form an n-dimensional version of $J_1(\delta)$ on the slice.

Proof: The slice $R^n \times \{\delta\ell\}$ is triangulated by the collection of n-simplices formed by taking the first $n+1$ vertices of members (y,ψ,a) of $J_1(\delta)$ with $y(0) = \delta\ell$, $\psi(n) = 0$, and $a(n) = 1$. Ignoring the 0-th coordinate, these n-simplices satisfy all membership criteria for $J_1(\delta)$ in n-dimensions, and exhaust the set of simplices that do so. $\square$

Sections of the triangulation $J_1(\delta)$ provide manifold blocks of uniform mesh. The particular sections used to construct triangulations in D_1 will now be defined.

4.2.3. DEFINITION. Define $J_1(\delta,\ell)$ for $\ell > 1$ and odd to be the set of all $(n+1)$-simplices in $J_1(\delta)$ which meet $R^n \times (\delta, \delta\ell)$.

4.2.4. PROPOSITION. $J_1(\delta,\ell)$ triangulates $R^n \times [\delta, \delta\ell]$.

Proof: Since $J_1(\delta)$ triangulates R^{n+1}, the faces of all simplices in $J_1(\delta,\ell)$ cover $R^n \times (\delta, \delta\ell)$ and are disjoint. Consider the simplices (y,ψ,a) in $J_1(\delta)$ with $\psi(n) = 0$ and either $y(0) = \delta$ and $a(n) = 1$, or $y(0) = \delta\ell$ and $a(n) = -1$. These simplices belong to $J_1(\delta,\ell)$ and cover $R^n \times \{\delta\}$ and $R^n \times \{\delta\ell\}$, respectively. Hence $R^n \times [\delta, \delta\ell]$ is covered by simplices in $J_1(\delta,\ell)$.

Any point in R^{n+1} outside of $R^n \times [\delta, \delta\ell]$ must lie in a simplex (y,ψ,a) of $J_1(\delta)$ satisfying either $y(0) < \delta$, $y(0) > \delta\ell$, $y(0) = \delta$ and $a(\psi^{-1}(0)) = -1$, or $y(0) = \delta\ell$ and $a(\psi^{-1}(0)) = 1$. Simplices satisfying any of these conditions miss $R^n \times (\delta, \delta\ell)$ and hence cannot belong to $J_1(\delta,\ell)$. Therefore $J_1(\delta,\ell)$ covers $R^n \times [\delta, \delta\ell]$ precisely. □

In light of Proposition 4.2.2, the upper and lower boundaries of $R^n \times [\delta, \delta\ell]$ inherit n-dimensional versions of $J_1(\delta)$ from $J_1(\delta,\ell)$.

The next components of D_1 to be specified are the blocks with expanding and refining mesh. Both of these are derived from Todd's refining triangulation J_3. Let $J_3^0 = \{y \in R^{n+1} : y(0) = 2^{-k}$ for $k \in Z_+$ and $y(i)/y(0)$ integral for $1 \le i \le n\}$ be the set of vertices and $J_3^{0c} = \{y \in J_3^0 : y(i)/y(0)$ is odd for $1 \le i \le n\}$ the set of central vertices. To each central vertex y with $y(0) = 2^{-k}$ and $k \ge 1$, there corresponds a closest central vertex z with $z(0) = 2^{1-k}$. The vertex z may be represented as $z = y - y(0)\nu$, where $\nu \in A^{n+1}$ and $\nu(i) = -1$ or $+1$ according as $y(i)/y(0)$ equals 1 or 3 (mod 4).

4.2.5. DEFINITION. J_3 consists of all $(n+1)$-simplices $\tau = (y_{-1}, y_0, \ldots, y_n)$ such that for some triple (y, ψ, a) in $J_3^{0c} \times \Psi_{n+1} \times A^{n+1}$ with $y(0) \leq \frac{1}{2}$,

$$y_{-1} = y$$

$$y_i = y_{i-1} + y(0)\ a(i)\ e_{\psi(i)}\ , \qquad \text{for } 0 \leq i < j \equiv \psi^{-1}(0)$$

$$y_j = y_{j-1} - y(0) \sum_{\ell=j}^{n} \nu(\psi(\ell))\ e_{\psi(\ell)}$$

$$y_i = y_{i-1} + 2y(0)\ \nu(\psi(i))\ e_{\psi(i)}\ , \qquad \text{for } j < i \leq n,$$

where ν is as above.

The simplex τ is also written $\tau = (y, \psi, a)$. Note that only the first $j-1$ components of the vector a are used in the definition. In Lemma 5.2 of [20], Todd proves that J_3 triangulates $R^n \times (0,1]$.

The triangulation J_3 possesses a hereditary property similar to $J_1(\delta)$.

4.2.6. PROPOSITION. The faces of simplices in J_3 which meet the slice $R^n \times \{2^{-\ell}\}$ for $\ell \in Z_+$ form an n-dimensional version of $J_1(2^{-\ell})$ on the slice.

Proof: For $\ell \geq 1$ the slice $R^n \times \{2^{-\ell}\}$ is triangulated by the collection of n-simplices formed by taking the first n+1 vertices of members (y,ψ,a) of J_3 with $y(0) = 2^{-\ell}$ and $\psi(n) = 0$. Ignoring the 0-th coordinate, these simplices satisfy all membership criteria for $J_1(2^{-\ell})$ in n-dimensions and are the only ones that do so. An equivalent way to represent these n-simplices is to take the last n+1 vertices of members (y,ψ,a) of J_3 with $y(0) = 2^{-\ell-1}$ and $\psi(0) = 0$. This representation covers the case $\ell = 0$. □

The 0-th coordinate of the central vertex of simplices in J_3 plays a role similar to the scale factor δ in $J_1(\delta)$. Later in this section the manifold blocks extracted from J_3 will be translated vertically, and hence the defining recursions in 4.2.5 will no longer hold. If, however, one replaces $y(0)$ with the appropriate scale factor δ, the recursions will still be valid. To cope with this eventuality, simplices in vertical translates of J_3 will be denoted (y,ψ,a,δ), where δ is an appropriate scale factor 2^{-k}. For consistency simplices in vertical translates of $J_1(\delta,\ell)$ will likewise be denoted (y,ψ,a,δ).

The sections of J_3 which provide manifold blocks of expanding mesh will now be defined.

4.2.7. DEFINITION. Define $J_3(\ell_1, \ell_2)$ for integers $\ell_2 > \ell_1 \geq 0$ to be the set of all (n+1)-simplices in J_3 which meet $R^n \times (2^{-\ell_2}, 2^{-\ell_1})$.

4.2.8. **PROPOSITION.** $J_3(\ell_1, \ell_2)$ triangulates $R^n \times [2^{-\ell_2}, 2^{-\ell_1}]$.

Proof: Since J_3 triangulates $R^n \times (0, 1]$, the faces of simplices in $J_3(\ell_1, \ell_2)$ cover $R^n \times (2^{-\ell_2}, 2^{-\ell_1})$ and are disjoint. Every simplex (y,ψ,a) in J_3 with $y(0) = 2^{-\ell_1 - 1}$ and $\psi(0) = 0$ belongs to $J_3(\ell_1,\ell_2)$, and these simplices cover $R^n \times \{2^{-\ell_1}\}$. Similarly every simplex (y,ψ,a) in J_3 with $y(0) = 2^{-\ell_2}$ and $\psi(n) = 0$ belongs to $J_3(\ell_1, \ell_2)$, and these simplices cover $R^n \times \{2^{-\ell_2}\}$. Hence $J_3(\ell_1, \ell_2)$ covers $R^n \times [2^{-\ell_2}, 2^{-\ell_1}]$.

Any point in $R^n \times (0,1]$ lying outside of $R^n \times [2^{-\ell_2}, 2^{-\ell_1}]$ must lie in a simplex (y,ψ,a) of J_3 satisfying either $y(0) \geq 2^{-\ell_1}$ or $y(0) < 2^{-\ell_2}$. Since no such simplex meets $R^n \times (2^{-\ell_2}, 2^{-\ell_1})$, none can belong to $J_3(\ell_1, \ell_2)$. Hence $J_3(\ell_1, \ell_2)$ covers $R^n \times [2^{-\ell_2}, 2^{-\ell_1}]$ precisely. □

In light of Proposition 4.2.6 the upper and lower boundaries of $R^n \times [2^{-\ell_2}, 2^{-\ell_1}]$ inherit n-dimensional versions of $J_1(2^{-\ell_1})$ and $J_1(2^{-\ell_2})$, respectively, from $J_3(\ell_1, \ell_2)$.

The only primary building block that remains to be specified is the one with refining mesh. It is obtained by merely inverting the expanding segment.

4.2.9. DEFINITION. Define $-J_3(\ell_1, \ell_2)$ for integers $\ell_2 > \ell_1 \geq 0$ to be the image of $J_3(\ell_1, \ell_2)$ under the linear homeomorphism that reverses the sign of the 0-th coordinate of points in R^{n+1}.

Since triangulations are preserved under linear homeomorphisms, the simplices in $-J_3(\ell_1, \ell_2)$ triangulate $R^n \times [-2^{-\ell_1}, -2^{-\ell_2}]$. Also, since the homeomorphism used to define $-J_3(\ell_1, \ell_2)$ affects only the 0-th coordinate, the n-dimensional versions of $J_1(2^{-\ell_i})$ on $R^n \times \{2^{-\ell_i}\}$ for $i = 1,2$ get transferred intact to the boundary hyperplenes of $R^n \times [-2^{-\ell_1}, -2^{-\ell_2}]$.

Now that all the necessary building blocks have been defined, the next step in the specification of D_1 is to explain how these blocks fit together to form triangulations of $R^n \times [0, \infty)$. Each member of D_1 is characterized by a sequence $\langle B^k \rangle$ of blocks of simplices, a sequence $\langle t^k \rangle$ of block interface heights, and a sequence $\langle \delta^k \rangle$ of block interface scale factors. The simplices in each block B^k triangulate the slab $R^n \times [t^{k-1}, t^k]$. The blocks are configured so that both B^k and B^{k+1} induce an n-dimensional version of $J_1(\delta^k)$ on the interface $R^n \times \{t^k\}$. Each B^k is a vertical translate of either $J_1(\delta, \ell)$, $J_3(\ell_1, \ell_2)$, or $-J_3(\ell_1, \ell_2)$. Blocks with odd sequence numbers are translates of $J_1(\delta, \ell)$, while even-numbered blocks may be translates of either $J_3(\ell_1, \ell_2)$ or $-J_3(\ell_1, \ell_2)$. Each block B^k is assigned the type code $\beta^k = -1, 0,$ or 1 according as its mesh is refining, uniform, or expanding, i.e., according as the block is a vertical translate of $-J_3(\ell_1, \ell_2)$, $J_1(\delta, \ell)$, or $J_3(\ell_1, \ell_2)$.

In practice the blocks are specified one at a time, each depending on its predecessors and the requirements of the economic algorithm. The dynamic construction procedure is really, therefore, just an induction scheme for defining new blocks in terms of previous ones. This scheme serves as the basis for the formal definition of the family D_1.

4.2.10. DEFINITION. The family D_1 of triangulations of $R^n \times [0,\infty)$ consists of all collections $\bigcup_{k=0}^{\infty} B^k$ of $(n+1)$-simplices taken from sequences $\langle B^k, t^k, \delta^k \rangle$ generated according to the following rules:

$k = 0$: $B^0 = -J_3(0,\ell) + e_0$ for some integer $\ell > 0$;
$t^0 = 1 - 2^{-\ell}$;
$\delta^0 = 2^{-\ell}$;

$k > 0$ and odd: $B^k = J_1(\delta^{k-1}, \ell) + (t^{k-1} - \delta^{k-1})e_0$ for some odd integer $\ell \geq 3$;
$t^k = t^{k-1} + (\ell-1)\,\delta^{k-1}$;
$\delta^k = \delta^{k-1}$;

$k > 0$ and even: Either
$B^k = J_3(\ell, -\log_2 \delta^{k-1}) + (t^{k-1} - \delta^{k-1})e_0$ for some non-negative integer $\ell < -\log_2 \delta^{k-1}$;
$t^k = t^{k-1} + (2^{-\ell} - \delta^{k-1})$;
$\delta^k = 2^{-\ell}$;

or
$B^k = -J_3(-\log_2 \delta^{k-1}, \ell) + (t^{k-1} + \delta^{k-1})e_0$ for some integer $\ell > -\log_2 \delta^{k-1}$;
$t^k = t^{k-1} + \delta^{k-1} - 2^{-\ell}$;
$\delta^k = 2^{-\ell}$.

Note that δ^k is always a non-positive integral power of two, so the $\log_2$ terms are always integral. Members of D_1 will be denoted interchangeably by the sequence $\langle B^k, t^k, \delta^k \rangle$ and by $\bigcup_{k=0}^{\infty} B^k$.

4.2.11. PROPOSITION. Every member $\langle B^k, t^k, \delta^k\rangle$ of D_1 with $t^k \to \infty$ triangulates $R^n \times [0,\infty)$.

Proof: Since triangulations are preserved under translation, each block B^k triangulates the slab $R^n \times [t^{k-1}, t^k]^*$. Since $t^k \to \infty$, the faces of all $(n+1)$-simplices in $\bigcup_{k=0}^{\infty} B^k$ cover $R^n \times [0,\infty)$. It remains only to show that these faces are disjoint. Every face lies either in an open slab $R^n \times (t^{k-1}, t^k)$ or in a slice $R^n \times \{t^k\}$. The faces within open slabs are disjoint because they belong to a single block of simplices. The faces within a slice $R^n \times \{t^k\}$ are disjoint because the two blocks of simplices B^k and B^{k+1} each induce the same n-dimensional triangulation $J_1(\delta^k)$ on the slice, as may be verified from Definition 4.2.10 and Propositions 4.2.2 and 4.2.6. □

Henceforth it will be assumed that all members $\langle B^k, t^k, \delta^k\rangle$ of D_1 satisfy $t^k \to \infty$. Later, after the family D_2 has been introduced, it will be demonstrated that this assumption is always satisfied in practice.

The first few blocks of a typical triangulation of class D_1 are illustrated in Figure 4.2.1. The right triangles in the figure represent 2-simplices of the triangulation. The sequence $\langle B^k, t^k, \delta^k\rangle$ of block types, block interface heights, and interface scale factors associated with this triangulation is $(-1, \frac{7}{8}, \frac{1}{8})$, $(0, 1\frac{3}{8}, \frac{1}{8})$, $(1, 1\frac{1}{2}, \frac{1}{4})$, $(0, 2, \frac{1}{4})$, $(-1, 2\frac{3}{16}, \frac{1}{16})$, and $(0, ?, \frac{1}{16})$. (The ? signifies that B^5 is still under construction.)

$^* t^{-1} = 0$.

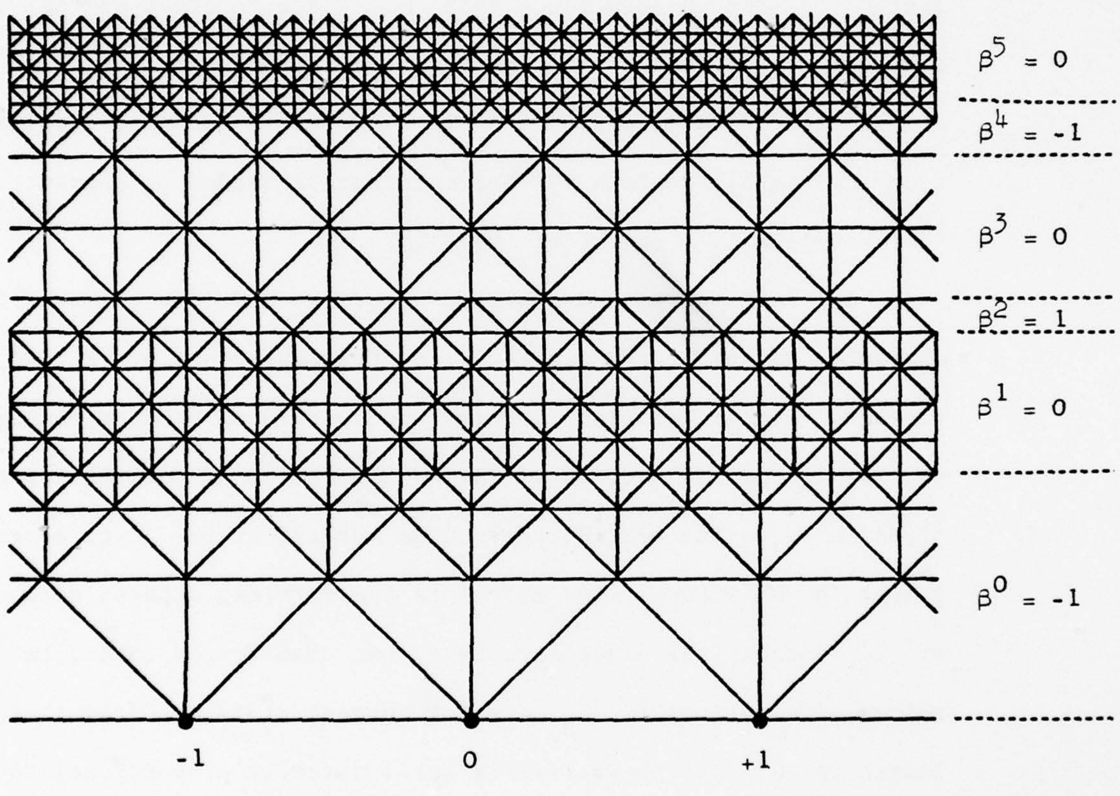

Figure 4.2.1. A triangulation of class D_1 on $R^1 \times [0,\infty)$

The inductive scheme presented in 4.2.10 suggests that each block B^k is defined in its entirety at step k of the induction. In practice the blocks are built-up one layer at a time as the algorithm climbs through $[0,\infty)$. Whenever a new maximum altitude is attained, a decision is made whether to extend the present block or begin a new one. In the latter instance the block interface height t^k, the interface scale factor δ^k, and the new block type β^{k+1} are recorded so that the manifold can be reconstructed should the algorithm turn back down. With these global parameters in place, the block number k and the local representation (y,ψ,a,δ) contain all the information needed to characterize any $(n+1)$-simplex in a member of D_1.

The family D_1 has now been described sufficiently well that the characterization of the class D_2 of triangulations of $S \times [0,\infty)$ can proceed. Little detailed work remains to be done because members of D_2 are merely images under an affine homeomorphism of portions of triangulations in D_1. The homeomorphism does not affect the 0-th coordinate of points in $R^n \times [0,\infty)$, and therefore the vertical aspects of the geometry of D_1, notably the block structure, get transferred intact to D_2. The shapes of simplices in D_2 are, of course, different from their pre-images in D_1, but their size is still directly proportional to the local scale factor δ. Thus the notions of refining, uniform, and expanding manifold blocks remain valid for D_2.

The first step in the definition of D_2 is to identify a subset of $R^n \times [0,\infty)$ that is triangulated by members of D_1 and is affinely homeomorphic to $S \times [0,\infty)$. Let

$$C \times [0,\infty) = \{x \in R^{n+1} : x(0) \geq 0 \quad \text{and} \quad 1 \geq x(1) \geq \cdots \geq x(n) \geq 0\} .$$

As elsewhere in this section the $[0,\infty)$ factor lies along the 0-th coordinate axis. According to Lemmas 4.1 and 5.5 of [20] and Proposition 4.2.11, the $(n+1)$-simplices in any triangulation of class D_1 which intersect the interior of $C \times [0,\infty)$ triangulate its closure. Furthermore, $C \times [0,\infty)$ is homeomorphic to $S \times [0,\infty)$ under the affine mapping $u : C \times [0,\infty) \to S \times [0,\infty)$ defined by $u(x) = Ux + e_1$, where U is the $(n+2) \times (n+1)$ matrix

$$\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 0 & \cdots & 0 \\ 0 & 1 & -1 & \cdots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & & \ddots & -1 \\ 0 & 0 & & \cdots & 1 \end{bmatrix}$$

The stage is now set for the formal definition of D_2.

4.2.12. DEFINITION. The family D_2 of triangulations of $S \times [0,\infty)$ consists of the images under u of all triangulations of $C \times [0,\infty)$ induced by members of D_1.

Members of D_2 do in fact triangulate $S \times [0,\infty)$ because triangulations are preserved under affine homeomorphisms. One may recall, however, that full coverage of $R^n \times [0,\infty)$ and hence $S \times [0,\infty)$ is only assured by the assumption $t^k \to \infty$. Fortunately this assumption always holds in practice, as will now be demonstrated.

4.2.13. PROPOSITION. If the sequence $\langle \delta^k \rangle$ in Definition 4.2.10 is bounded away from zero, then $t^k \to \infty$.

<u>Proof:</u> According to the definition of t^k for $k > 0$, either

$$t^k = t^{k-1} + (\ell-1)\,\delta^{k-1} \geq t^{k-1} + \delta^{k-1},$$

$$t^k = t^{k-1} + (2^{-\ell} - \delta^{k-1}) \geq t^{k-1} + \delta^{k-1},$$

or

$$t^k = t^{k-1} + \delta^{k-1} - 2^{-\ell} \geq t^{k-1} + \delta^k. \qquad \square$$

In light of Theorem 3.5.4, domain error must be reduced only so far in order for range error in an approximate equilibrium graph to meet specified tolerances. By accepting a positive level of range error, one effectively establishes a positive lower bound on the maximum acceptable domain error. Since the dynamic construction procedure always seeks the maximum acceptable domain error, the scale factors δ^k which define domain error may be (and henceforth will be) considered bounded away from zero. The claim follows from the preceding proposition.

Simplices comprising members of D_2 admit the same representation (y,ψ,a,δ) as those in D_1, i.e., y is a central vertex, $\psi \in \Psi_{n+1}$, $a \in A^{n+1}$, and δ is a scale factor. The central vertices here are the images under u of central vertices of corresponding simplices in D_1. The remaining vertices satisfy the same recursions (4.2.1 and 4.2.5) as their counterparts in D_1 provided the canonical unit vectors e_j are replaced by the columns u_j of U for $0 \leq j \leq n$. As noted earlier, the term $y(0)$ must be replaced by δ in 4.2.5. The formula for computing ν, whose image $U\nu$ now points toward the central vertex y from the closest central vertex z in the next coarsest layer of an expanding or refining manifold block, becomes more complicated because of the transformation U. The components of ν for simplices in D_2 are given by $\nu(n) = -1$ or $+1$ according as $y(n)/\delta$ is 1 or 3 (mod 4) and $\nu(i) = \nu(i+1)$ or $-\nu(i+1)$ according as $y(i)/\delta$ is 0 or 2 (mod 4) for $1 \leq i < n$. The initial component $\nu(0)$ is $+1$ or -1 according as the manifold block is refining (Type -1) or expanding (Type 1).

Throughout this section the terms triangulation and manifold have been used interchangeably. Strictly speaking, a manifold is a slightly more general structure. For practical purposes, however, manifolds may be considered to be induced by triangulations in the following manner.

4.2.14. PROPOSITION. Every triangulation in D_2 induces an abstract pseudomanifold K^{n+1} on $S \times [0,\infty)$ whose abstract $(n+1)$-simplices consist of the vertex sets of $(n+1)$-simplices in the triangulation.

Proof: Consider any triangulation in D_2 and the induced collection K^{n+1} of abstract $(n+1)$-simplices. Let $\sigma = \{v_0, \ldots, v_n\}$ be an abstract n-simplex in K^n. By definition σ is the vertex set of a facet of some $(n+1)$-simplex $\tau = \operatorname{conv}(\sigma \cup \{v_{n+1}\})$ in the triangulation. Suppose $\operatorname{conv} \sigma$ is also a facet of another $(n+1)$-simplex $\mu = \operatorname{conv}(\sigma \cup \{u_{n+1}\})$ in the triangulation. Since $\operatorname{aff} \sigma$ is an n-dimensional hyperplane which cuts the $(n+1)$-dimensional polyhedron $S \times [0,\infty)$, and since the interiors of τ and μ are disjoint, then v_{n+1} and u_{n+1} must lie on opposite sides of $\operatorname{aff} \sigma$. Clearly τ and μ are the only $(n+1)$-simplices of the triangulation which can contain $\operatorname{conv} \sigma$. Furthermore, if σ lies in a facet of $S \times [0,\infty)$, then τ is the only $(n+1)$-simplex containing σ, since no other simplex of the triangulation could lie on the opposite side of $\operatorname{aff} \sigma$ from v_{n+1} and still reside in $S \times [0,\infty)$. Hence conditions 2.2.1(b) and (c) in the definition of abstract pseudomanifold are satisfied.

Condition 2.2.1(a) is also satisfied because $S \times \{0\}$ is the image under u of $\{x \in C \times [0,\infty) : x(0) = 0\}$, which is a facet of the $(n+1)$-simplex in $B^0 = -J_3(0,\ell) + e_0$ obtained from the $(n+1)$-simplex (y,ψ,a) in J_3 with $y = (\frac{1}{2}, \ldots, \frac{1}{2})$ and $\psi = (0, n, n-1, \ldots, 1)$. Finally since the interface scale factors δ^k of the triangulation are considered to be bounded away from zero, only a finite number of $(n+1)$-simplices can lie below any given level in $S \times [0,\infty)$. Therefore K^{n+1} satisfies all the requirements of Definition 2.2.1. $\square$

The definition and analysis of the dynamically defined families of manifolds D_1 and D_2 are now complete. The discussion of these structures will be concluded with the specification of their pivot rules. These are formulae for determining which $(n+1)$-simplex in the manifold shares a particular facet with a given $(n+1)$-simplex. Hence they facilitate calculation of the incoming vertex in the fundamental algorithm. The pivot rules for D_1 and D_2 are hierarchically organized into two tiers corresponding to the global and local levels of the characterization of simplices. The global tier determines whether the new simplex belongs to the current or an adjacent manifold block, and pinpoints a set of detailed formulae in the local tier. The local formulae are then applied to obtain an explicit representation for the new simplex.

Any simplex τ in a triangulation of class D_1 or D_2 can be fully characterized by the global parameter k (block number) and the local parameters (y, ψ, a, δ). Alternatively τ may be expressed directly as $(y_{-1}, y_0, y_1, \ldots, y_n)$. This latter representation is used to identify the dropping vertex y_i. The new simplex τ' which shares all of τ's vertices except y_i will be denoted (y', ψ', a', δ'). To determine the parameters of τ', one first consults Table 4.2.2 to discover which manifold block k' contains τ' and which set of detailed formulae to use in the next step. One then refers to the indicated set of local pivot rules to obtain expressions for (y', ψ', a', δ'). In the event that the local rules are found in Table 4.2.3, the criteria $i = -1$ and $i = j-1$, or $i = n$ and $i = j$ may hold simultaneously. The first criterion in each pair should be used in these instances.

TABLE 4.2.2. Global Pivot Rules for D_1 and D_2

					k'	Local Table Reference
$\beta^k = -1$	$y(0) = t^k$ *	$i = n$	$\psi(n) = 0$		$k+1$	4.2.6 $(-1 \uparrow 0)$
	$y(0) = t^{k-1} + \delta$	$i = -1$	$\psi(0) = 0$		$k-1$	4.2.6 $(-1 \downarrow 0)$
	otherwise				k	4.2.3
$\beta^k = 0$	$y(0) = t^k$ *	$i = n$	$\psi(n) = 0$	$\beta^{k+1} = -1$	$k+1$	4.2.6 $(0 \uparrow -1)$
			$a(n) = -1$	$\beta^{k+1} = 1$	$k+1$	4.2.6 $(0 \uparrow 1)$
	$y(0) = t^{k-1}$	$i = n$	$\psi(n) = 0$	$\beta^{k-1} = -1$	$k-1$	4.2.6 $(0 \downarrow -1)$
			$a(n) = 1$	$\beta^{k-1} = 1$	$k-1$	4.2.6 $(0 \downarrow 1)$
	otherwise				k	4.2.4
$\beta^k = 1$	$y(0) = t^k - \delta$ *	$i = -1$	$\psi(0) = 0$		$k+1$	4.2.6 $(1 \uparrow 0)$
	$y(0) = t^{k-1}$	$i = n$	$\psi(n) = 0$		$k-1$	4.2.6 $(1 \downarrow 0)$
	otherwise				k	4.2.5

*If B^k is latest block, then $t^k \equiv \infty$.

TABLE 4.2.5. Local Pivot Rules for Type -1 Block -- D_2

		y'	δ'	ψ'	a' [2]	ν' [2]
$i = -1$	$\psi(0) = 0$	$y - \delta u_0 - \delta \sum_{\ell=1}^{n} \nu(\ell)\, u_\ell$	2δ	$[\psi(1), \ldots, \psi(n), \psi(0)]$	$[\nu(\psi(1)), \ldots, \nu(\psi(n)), 1]$	Use defining formula for Type -1 block.
	$\psi(0) > 0$	$y + 2\delta\, a(0)\, u_{\psi(0)}$	δ	ψ	$[-a(0), a(1), \ldots, a(n)]$	$\nu - 2\nu(\psi(0))\, e_{\psi(0)}$
$0 \le i < j-1$ [1]		y	δ	$[\psi(0),\ldots,\psi(i+1),\psi(i),\ldots,\psi(n)]$	$[a(0), \ldots, a(i+1), a(i), \ldots, a(n)]$	ν
$i = j-1$	$a(j-1) = \nu(\psi(j-1))$	y	δ	$[\psi(0),\ldots,\psi(j),\psi(j-1),\ldots,\psi(n)]$	$[a(0), \ldots, a(j-2), 1, \ldots, 1]$	ν
	$a(j-1) = -\nu(\psi(j-1))$	y	δ	$[\psi(0),\ldots,\psi(j-2),\psi(j),\ldots,\psi(n),\psi(j-1)]$	$[a(0), \ldots, a(j-2), 1, \ldots, 1]$	ν
$i = j$		y	δ	$[\psi(0),\ldots,\psi(j+1),\psi(j),\ldots,\psi(n)]$	$[a(0), \ldots, a(j-1), \nu(\psi(j+1)), 1, \ldots, 1]$	ν
$j < i < n$		y	δ	$[\psi(0),\ldots,\psi(i+1),\psi(i),\ldots,\psi(n)]$	a	ν
$i = n$	$\psi(n) = 0$	$y + \delta/2\, u_0 + \frac{\delta}{2} \sum_{\ell=0}^{n-1} a(\ell)\, u_{\psi(\ell)}$	$\frac{\delta}{2}$	$[\psi(n),\psi(0),\ldots,\psi(n-1)]$	$[1, \ldots, 1]$	$[1, a(\psi^{-1}(1)), \ldots, a(\psi^{-1}(n))]$
	$\psi(n) > 0$	y	δ	$[\psi(0),\ldots,\psi(j-1),\psi(n),\psi(j),\ldots,\psi(n-1)]$	$[a(0),\ldots,a(j-1),-\nu(\psi(n)),1,\ldots,1]$	ν

1) $j \equiv \psi^{-1}(0)$ 2) Unused components of a' and ν' are filled with ones.

TABLE 4.2.4. Local Pivot Rules for Type 0 Block -- D_2

	y'	δ'	ψ'	a'
$i = -1$	$y + 2\delta\, a(0)\, u_{\psi(0)}$	δ	ψ	$[-a(0), a(1), \ldots, a(n)]$
$0 \leq i < n$	y	δ	$[\psi(0), \ldots, \psi(i+1), \psi(i), \ldots, \psi(n)]$	$[a(0), \ldots, a(i+1), a(i), \ldots, a(n)]$
$i = n$	y	δ	ψ	$[a(0), \ldots, a(n-1), -a(n)]$

TABLE 4.2.5. Local Pivot Rules for Type 1 Block -- D_2

		y'	ν'	δ', ψ', a'
$i = -1$	$\psi(0) = 0$	$y + \delta u_0 - \delta \sum_{\ell=1}^{n} \nu(\ell)\, u_\ell$	Use defining formula for Type 1 block.	Refer to Table 4.2.3 for these values.
	$\psi(0) > 0$	$y + 2\delta\, a(0)\, u_{\psi(0)}$	$\nu - 2\nu(\psi(0))\, e_{\psi(0)}$	
$0 \leq i < n$		Refer to Table 4.2.3 for these cases.		
$i = n$	$\psi(n) = 0$	$y - \delta/2\, u_0 + \frac{\delta}{2} \sum_{\ell=0}^{n-1} a(\ell)\, u_{\psi(\ell)}$	$[-1, a(\psi^{-1}(1)), \ldots, a(\psi^{-1}(n))]$	Refer to Table 4.2.3 for these values.
	$\psi(n) > 0$	y	ν	

TABLE 4.2.6. Local Pivot Rules for Interblock Transitions -- D_2

	y'	δ'	ψ'	a' *	ν' *
$-1 \uparrow 0$	y	δ	ψ	$[a(0),\ldots,a(n-1),1]$	Not used.
$0 \uparrow -1$	$y + \delta/2\, u_0 + \frac{\delta}{2} \sum_{\ell=0}^{n-1} a(\ell)\, u_{\psi(\ell)}$	$\frac{\delta}{2}$	$[\psi(n), \psi(0), \ldots, \psi(n-1)]$	$[1, \ldots, 1]$	$[1, a(\psi^{-1}(1)), \ldots, a(\psi^{-1}(n))]$
$0 \uparrow 1$	y	δ	ψ	$[a(0),\ldots,a(n-1),1]$	Use defining formula for Type 1 block.
$1 \uparrow 0$	$y + \delta u_0 - \delta \sum_{\ell=1}^{n} \nu(\ell)\, u_\ell$	2δ	$[\psi(1), \ldots, \psi(n), \psi(0)]$	$[\nu(\psi(1)), \ldots, \nu(\psi(n)), 1]$	Not used.
$-1 \downarrow 0$	$y - \delta u_0 - \delta \sum_{\ell=1}^{n} \nu(\ell)\, u_\ell$	2δ	$[\psi(1), \ldots, \psi(n), \psi(0)]$	$[\nu(\psi(1)), \ldots, \nu(\psi(n)), -1]$	Not used.
$0 \downarrow -1$	y	δ	ψ	$[a(0),\ldots,a(n-1),1]$	Use defining formula for Type -1 block.
$0 \downarrow 1$	$y - \delta/2\, u_0 + \delta/2 \sum_{\ell=0}^{n-1} a(\ell) u_{\psi(\ell)}$	$\frac{\delta}{2}$	$[\psi(n), \psi(0), \ldots, \psi(n-1)]$	$[1, \ldots, 1]$	$[-1, a(\psi^{-1}(1)), \ldots, a(\psi^{-1}(n))]$
$1 \downarrow 0$	y	δ	ψ	$[a(0),\ldots,a(n-1),-1]$	Not used.

*Unused components of a' and ν' are filled with ones.

Also, in Type -1 and Type 1 blocks the central vertex pointer ν can frequently be updated instead of computed from the definition. Entries for ν and its successor ν' have been included in the tables where appropriate.

The local pivot tables are currently set up for triangulations of class D_2. They can easily be converted to D_1, however, by replacing the column vectors u_j (taken from the matrix U) by the canonical unit vectors e_j of R^{n+1} for $0 \leq j \leq n$. Also, the central vertex pointer ν must be computed by different formulae (given earlier) according to whether the manifold belongs to D_1 or D_2.

In closing it is worth mentioning that the local pivot rules in Tables 4.2.3 - 4.2.6 are really not intended for implementation on a computer in their present form. Highly efficient but cumbersome variations of these rules are available which generate the incoming vertex by operating on only one or two existing vertices of a simplex. These variations, which are not shown here because of their complexity, were incorporated in the computer programs developed for this study.

4.3. Error Control Heuristics

In order for dynamic manifold construction to achieve the control over range error for which it was intended, external requests for particular block types must be supplied to the construction routines. These requests are honored each time a new layer is added to the manifold. The generation of requests naturally occurs in two steps. First the current approximate equilibrium must be analyzed to determine its range error, and second a decision based on the findings must be made regarding which request to issue. Both the analysis and decision steps rely heavily on heuristics.

The measurement of range error requires heuristics because the actual consumption and production plans guaranteed to exist by Theorem 3.5.4 cannot be computed in practice. Actual plans are the standards against which approximate equilibria ought to be judged. In their place plausible surrogates must be used. To this end the pseudo-production plan is treated as if it were actual, and the demand point used to label the designated vertex $(\pi, r, t) \in \mathcal{g}(\sigma)$ is taken to be the actual consumption plan. (For single-valued demand functions this latter assignment is precise.) The profitability components of range error then become the after-tax profitabilities at prices and revenue levels (π, r) of the unit activities comprising the pseudo-production plan (these are ideally zero). The supply-demand components become the differences between pseudo-supply and the assigned demand values. Tax-revenue components are similarly measured in terms of r, pseudo-producer taxes, and consumer taxes at the assigned demand point. To allow for disparities among the units used to measure commodity flows, supply-demand errors are calculated as a percent of total demand.

The rules for converting range error data into control signals must also be based on heuristics because of certain practical limitations of the manifold construction process. Chief among these is the fact that grid size cannot be changed instantaneously, even when the algorithm is moving upward through new regions of the cylinder. This constraint exists because block interfaces can only occur at certain discrete levels and because refining and expanding blocks must be separated by uniform blocks. Also, the data structures used to store global manifold parameters would take up too much space if blocks were switched too often. To get around these limitations, range error deviations must be detected well before critical tolerances are reached. Also, corrective action must be strong enough to restore errors to a level where they are likely to stay put for a while.

Another difficulty is that range error consists of many components, each with its own freedom of movement, while the control mechanism has only one degree of freedom -- grid size. Hence the decision rules must respond to that combination of error components in each situation most likely to induce the entire body of components to move as a whole.

To implement these ideas a control system based on three tolerance levels -- loose, central, and tight -- was devised. The central tolerance represents the most desirable level of range error. The loose and tight tolerances denote, respectively, the maximum and minimum permissible values. Different settings of the three levels can be made for different error components. The computer programs, for example, use one set of tolerances for profitability errors and another for supply-demand errors.

Whenever selected components of range error stray beyond the loose or tight tolerances, a manifold refinement or expansion is requested until specific components are brought back to the central tolerance level. The control mechanism responds to any error component that violates a loose tolerance, since the validity of the approximation is jeopardized by such deviations. Tight tolerances, on the other hand, trigger corrections only when all error components cross them. Asymmetry between the response criteria is necessary to prevent both sets of tolerances from being violated simultaneously.

The detailed decision rules used to generate manifold type requests are shown in Table 4.3.1. All cases presume that a demand-labeled vertex exists. If not, then the indicated request is overridden by a Type -1 request (or Type 0 if the current block is Type 1).

TABLE 4.3.1. Manifold Request Decision Rules

Current Block Type	Status of Range Error Components	Requested Block Type
-1	Some above central tolerances	-1
	All below central tolerances	0
0	Some above loose tolerances	-1
	All below tight tolerances	1
	Otherwise	0
1	Some above central tolerances	0
	All below central tolerances	1

4.4. Structure of Computer Programs

A flow diagram depicting the major processing sections of the computer programs used to implement the economic algorithm appears in Figure 4.4.1. The diagram concisely summarizes the logic of the algorithm and serves as an introduction to the software employed in the numerical experiments of Chapter 6. All important analytical functions are discharged in the sections entitled manifold pivot, label generation, label system pivot, and tolerance checking. The remaining sections merely provide administrative support.

The data input and initialization section supplies the program with two kinds of parameters -- economic and operational. The latter group includes tolerance levels, basis re-inversion frequency, and equilibrium report frequency. Main program data structures such as the right hand side p of the label systems and the demand offset function $c(\cdot)$ also get initialized in this section. Data structures controlled by subroutines get initialized during the first subroutine call. The final task performed in the initialization section is to print out all input parameters for purposes of verification.

The manifold pivot section maintains a representation of the current $(n+1)$-simplex $\tau^k \in K^{n+1}$. Upon receipt of the position of a dropping vertex from the label system pivot section, a new $(n+1)$-simplex τ^{k+1} is generated according to the pivot rules in Tables 4.2.2 - 4.2.6. The incoming vertex v^{k+1} gets passed to the label generation section. In the process of computing v^{k+1} the program attempts to honor the latest manifold type request issued by the tolerance checking section. This may

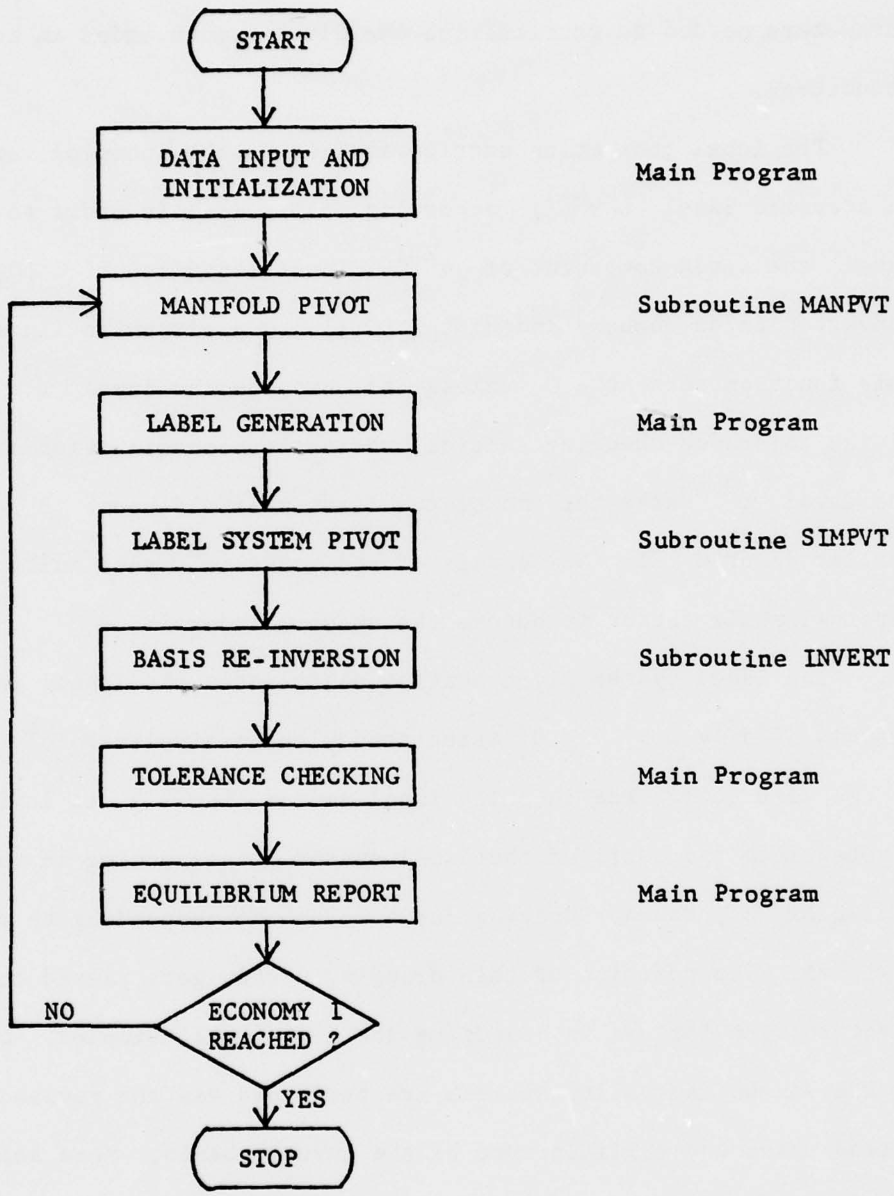

Figure 4.4.1. General flow diagram for programs implementing economic algorithm.

entail the creation of a new block of simplices. If so, then the global parameters needed to characterize the block are recorded in resident data structures.

The label generation section assigns to the incoming vertex v^{k+1} an economic label $L(v^{k+1})$ according to 3.3.12. In order to generate the label, the first component of v^{k+1} (interpreted as $t' \in [0,\infty)$) gets converted to an economy index $t \in [0,1]$ by a piecewise linear function. This function sets $t = 0$ unless t' exceeds the level t^0 (determined by the tolerance checking section) where the economic deformation begins. The level t^0 marks the end of the first manifold block B^0 and was earlier denoted 2I. Any excess of t' over t^0 gets multiplied by a vertical scale factor to become the economy index t.

The label system pivot section manipulates the linear inequality systems $L(\sigma^k)y = p$, $y \geq 0$ associated with n-simplices σ^k traversed by the algorithm. The incoming label vector $L(v^{k+1})$ is lexicographically pivoted into the basis of the label system corresponding to the current n-simplex σ^k, thereby driving out a column corresponding to one of σ^k's vertices. The position of this dropping vertex gets passed to the manifold pivot section for use in computing the next $(n+1)$-simplex. Operations on the linear inequality systems are performed via the revised simplex method using the explicit form of the inverse basis. More sophisticated basis-handling techniques such as LUD decomposition appear to promise only modest improvements because of the need for full lexicographic pivots.

The basis re-inversion section periodically inverts the label system matrix $L(\sigma^k)$ to obtain a more accurate inverse basis. The numerical technique used for inversion is Gauss-Jordan elimination with row rearrangement. Re-inversion occurs every N iterations, where N is an input parameter.

The tolerance checking section extracts from the label system $L(\sigma^k)y = p,\ y \geq 0$ an approximate equilibrium for the economy represented by some demand-labeled vertex of σ^k. Range error components for this approximation are computed one at a time until an unambiguous manifold request can be determined from Table 4.3.1. The request is then relayed to the manifold pivot section for consideration during generation of the next $(n+1)$-simplex. When the first satisfactory approximate equilibrium is found for the initial economy $\mathcal{E}(0)$, the tolerance checking section signals the label generation section to begin the economic deformation.

The equilibrium report section produces two kinds of printed reports. The first summarizes in a single line the current state of the manifold, the n-simplex σ^k, and the label system associated with σ^k. The frequency of this report is controlled by an input parameter. The second report lists all details of the current approximate equilibrium. It is produced each time the economy index changes by a pre-specified increment. Full equilibrium reports typically appear interspersed among the more numerous one-line summaries.

CHAPTER 5

EXAMPLES OF ECONOMIC DEFORMATIONS

The economic model of Chapter 3 was posed in terms of market aggregates possessing various abstract properties. This recondite approach was adopted to ease the notational burden in proofs and allow for maximum flexibility in applications. To justify the use of such abstractions, however, and provide insight into the range of situations covered, concrete examples are required. The purpose of the present chapter is to examine some typical microeconomic formulations which give rise under deformation to the sorts of aggregates assumed in Chapter 3.

The examples treated here include one detailed model of consumption and two of production. The consumption model features the usual assortment of consumers, consumption sets, preference orderings (reflected in utility functions), and initial endowments. In addition each consumer holds claims to revenue disbursements and pays taxes. Many of these components may be deformed to sweep out a family of economies. Upon aggregation the consumer-specific components yield a market demand correspondence $\Xi(\cdot)$, an initial endowments function $w(\cdot)$, and a tax function $\phi(\cdot)$ which satisfy all the requirements of Chapter 3.

The production models feature a finite group of sectors, each with its own set of non-slack unit activities and unit tax rates. These, too, may be deformed across the family of economies. The end product in each of the models is an aggregate activity correspondence $\mathcal{B}(\cdot)$ and a tax function $\gamma(\cdot)$ which satisfy all the conditions assumed in Chapter 3.

The examples covered in this chapter were selected because of their widespread use in empirical microeconomic studies. Consumer demands are derived from CES utility functions, and incur ad valorem consumption taxes of the type described in [15], [19]. One of the production models features the well-known activity analysis formulation of conversion technology. The other employs multi-factor CES production functions to describe input-output relationships in each sector. Both production models incorporate ad valorem production taxes of the type used by Shoven in [16], [17].

Many variations and extensions of these examples are possible within the framework of the general theory. Even more can be envisioned which deviate from the general theory only in their behavior on or near the vertical facets of $S \times T$. These, too, can often be solved by the economic algorithm. The examples considered below, however, suffice to describe all the numerical experiments reported in Chapter 6.

As in Chapter 3 the examples inhabit a space of $m+1$ commodities indexed $i = 0, \ldots, m$ and $n-m$ revenue systems indexed $i = m+1, \ldots, n$. Some of the symbols of Chapter 3, notably c, d, p, ϕ_j, and γ_j, are reused here with different meanings.

5.1. CES Consumption

The CES utility function originally appeared as a production function in a paper by Arrow, Chenery, Minhas, and Solow [1]. Many of its properties make it both appropriate and expedient for use in empirical models of consumer preferences, although certain deficiencies such as the absence of an income effect must be tolerated. The function is defined on the non-negative orthant of commodity space and assumes the basic form

$$(5.1.1) \qquad u(x) = \left[\sum_{i=0}^{m} a(i)^{1/b} x(i)^{(b-1)/b} \right]^{b/(b-1)} ,$$

where b is the elasticity of substitution and a is a vector of weighting factors (also called demand intensities). For values $0 < b < 1$ the function can only be defined on the boundary of R_+^{m+1} via limits from the interior. To insure that u is non-decreasing, the weighting vector a must be non-negative, and to exclude trivial cases only non-zero weighting vectors will be considered. Whenever $a(i) > 0$, good i will be described as "desired" by the consumer whose preferences are represented by u.

Many values of b are technically possible. However, for $b < 0$ the function is convex, which violates the principle of decreasing marginal utility. Negative values of b are, therefore, excluded. For $b > 0$ $(b \neq 1)$ the function is concave and hence admissible. (Both properties follow from the generalized Minkowski inequality in Section 1.17 of [3].) For the critical value $b = 1$, $u(x)$ is undefined in the above form. However, if $ea = 1$, then letting $b \to 1$ and applying L'Hospital's rule yields

$$(5.1.2) \qquad u(x) = \prod_{i=0}^{m} x(i)^{a(i)} ,$$

which is the familiar Cobb-Douglas utility function. This may be proven concave via the gradient inequality and the generalized arithmetic-geometric mean inequality appearing in Section 1.14 of [3].

Values of b between zero and one correspond to complementary goods while values of $b > 1$ connote substitutes. The extreme cases of perfect complements and perfect substitutes are covered by limiting forms of 5.1.1 as $b \to 0^+$ and $b \to \infty$, respectively. Only finite values of b will be admitted in models treated here. A concise summary of the properties of 5.1.1 for $0^+ \leq b \leq \infty$ appears in Section 3-6 of [10].

The reason for introducing the CES utility function is to determine how a rational consumer whose preferences are reflected in this function makes his purchase decisions. Suppose such a consumer has income $\mu \geq 0$ to spend for goods and services and must pay prices $p \in R_+^{m+1}$ for these items. His purchase decision problem is

$$
\begin{aligned}
(5.1.3) \qquad & \text{maximize} && u(x) \\
& \text{subject to} && px \leq \mu \\
& && x \in R_+^{m+1} \quad .
\end{aligned}
$$

This is a well-posed concave program, so either there exists an optimal solution ξ, or else the problem is unbounded. First note that if $a(i) = 0$, then u is independent of $x(i)$. Since $x(i)$ may cost money, it is always optimal to force $x(i) = 0$. The solution is then independent of $p(i)$.

Now consider the case where $a \gg 0$. Suppose all prices are positive. If $\mu = 0$ then the feasible region consists of the single point $x = 0$. If $\mu > 0$ then 5.1.3 is a bounded concave program over a feasible region with non-empty interior. Applying the Kuhn-Tucker sufficient conditions to 5.1.3 (with x restricted to the interior of R_+^{m+1} where u is differentiable) yields the unique optimal solution

$$(5.1.4) \qquad \xi = \left(\frac{a(0)}{p(0)^b}, \ldots, \frac{a(m)}{p(m)^b}\right) \frac{\mu}{\sum_{i=0}^{m} a(i)\, p(i)^{1-b}} .$$

Separate analyses are required for the cases $b \neq 1$ and $b = 1$ because of the different functional forms of u, but the outcome is the same.

In light of earlier remarks expression 5.1.4 also gives the optimal solution to 5.1.3 if $\mu = 0$, or if some of the $a(i) = 0$. It is even valid when the prices of undesired goods are zero, provided terms of the form $0/0$ are interpreted as zero. If, however, a desired good is free and the consumer has positive income, the purchase decision problem is unbounded (since $\partial u/\partial x(i) > 0$ for $x \gg 0$). For $b > 1$ such problems are unbounded even if $\mu = 0$.

Despite the fact that the consumer's purchases cannot be specified when desired goods are free, the economic model of Chapter 3 requires a market demand correspondence defined for all price combinations. The correspondence must satisfy certain technical conditions to insure that the algorithm behaves properly. To meet these technical requirements, artificial values are assigned to demand when desired goods are free.

The following proposition indirectly justifies these assignments by insuring that artificial values cannot be approached by demand labels.

5.1.5. **PROPOSITION.** The optimal solution ξ given in 5.1.4 diverges to $+\infty$ in norm as the prices of any desired goods approach zero from above, provided $\mu > 0$ and the other prices remain bounded.

Proof. Let $\langle p^k \rangle$ be an arbitrary sequence of prices converging to a limit p^∞ having zero components corresponding to desired goods. It suffices to show that $\langle p^k \rangle$ has a subsequence along which $\|\xi^k\| \to \infty$. Let $\beta = \{i : \text{good } i \text{ is desired and } p^\infty(i) = 0\}$. Since $\mu > 0$, it is enough to show that for some $i \in \beta$, $p^k(i)^b\, p^k(j)^{1-b} \to 0$ for all $j \in \beta$ along a subsequence. Suppose this does not occur. Then along every subsequence, for each $i \in \beta$ there exists $j \in \beta$ s.t. $p^k(i)^b\, p^k(j)^{1-b} \not\to 0$. Choose i_1 and i_2 from β s.t. $p^k(i_1)^b\, p^k(i_2)^{1-b} \not\to 0$, and then extract a subsequence along which $p^k(i_1)^b\, p^k(i_2)^{1-b} \geq \epsilon_{1,2} > 0$. Now choose $i_3 \in \beta$ s.t. $p^k(i_2)^b\, p^k(i_3)^{1-b} \not\to 0$ along the subsequence, and then extract a further subsequence along which $p^k(i_2)^b\, p^k(i_3)^{1-b} \geq \epsilon_{2,3} > 0$. Continue to build the sequence $i_1, i_2, i_3, \ldots$ until one of the i_j is repeated (this must occur since β is finite). Consider the segment $i_j, i_{j+1}, \ldots, i_\ell$ where $i_j = i_\ell$. Along the deepest subsequence

$$p^k(i_j)\ p^k(i_{j+1}) \cdots p^k(i_\ell)$$

$$= p^k(i_j)^b\ p^k(i_{j+1})^{1-b}\ p^k(i_{j+1})^b\ p^k(i_{j+2})^{1-b} \cdots p^k(i_{\ell-1})^b\ p^k(i_\ell)^{1-b}$$

$$\geq \epsilon_{j,j+1}\ \epsilon_{j+1,j+2} \cdots \epsilon_{\ell-1,\ell} > 0\ .$$

But this contradicts the fact that $p^k(i_j) \cdots p^k(i_\ell) \to 0$, thereby establishing the proposition. □

When $b \leq 1$ a direct calculation shows that $\xi^k(i) \to \infty$ for $i \in \beta$ regardless of the sequence $\langle p^k \rangle$. In this case the preceding argument is unnecessary. When $b > 1$, however, a given component $\xi^k(i)$ can be made to approach any positive limit or diverge to $+\infty$ by a suitable choice of $\langle p^k \rangle$. The proposition shows that regardless of the sequence $\langle p^k \rangle$, some component $\xi^k(i) \to +\infty$ for $i \in \beta$.

Enough preliminaries have now been established to permit the formal specification of the CES consumption model for the family $\{\mathcal{E}(t)\}_{t \in [0,1]}$. Each economy $\mathcal{E}(t)$ contains M groups of consumers indexed $j = 1, \ldots, M$. Each consumer group is characterized by four sets of numbers:

(a) Initial commodity holdings $w_j(t) \in R_+^{m+1}$;

(b) Revenue share factors $\rho_j(t) \in R_+^{n-m}$;

(c) A CES utility function of the form 5.1.1 with substitution elasticity $b_j(t) > 0$ and weighting factors $a_j^t \in R_+^{m+1} \backslash \{0\}$ (for $b_j(t) = 1$ the Cobb-Douglas form 5.1.2 with $ea_j^t = 1$ is presumed);

(d) A matrix $\Phi_j^t \in R_+^{(n-m)\times(m+1)}$ of ad valorem commodity tax rates. Each row corresponds to a revenue system and each column to a commodity.

The t-superscript (or argument) designates a consumer characteristic that can be deformed across the family of economies.

The initial endowments and revenue shares determine consumer income. At prices and revenue levels (π, r) consumer group j of economy $\mathcal{E}(t)$ has an income of $\pi w_j(t) + r\rho_j(t)$. This income is used to support consumption and pay taxes. Given a consumption pattern $x \in R_+^{m+1}$ the expenditure for good i is $\pi(i)\, x(i)$, and the tax levied by revenue system ℓ is $\Phi_j^t(\ell, i)\, \pi(i)\, x(i)$. Consequently the effective price paid for good i is $p_j^t(i) = (1 + e\Phi_j^t(\cdot, i))\, \pi(i)$. Faced with these effective prices the consumer deploys his income so as to maximize his satisfaction. The result is a demand response of the form 5.1.4, namely

$$(5.1.6) \qquad \xi_j(\pi, r, t) = (\xi_j(0; \pi, r, t), \ldots, \xi_j(m; \pi, r, t))$$

where

$$\xi_j(i; \pi, r, t) = \frac{a_j^t(i)}{p_j^t(i)^{b_j(t)}} \; \frac{\pi w_j(t) + r\rho_j(t)}{\sum_{\ell=0}^{m} a_j^t(\ell)\; p_j^t(\ell)^{1-b_j(t)}}$$

and

$$p_j^t(i) = (1 + e\Phi_j^t(\cdot, i))\, \pi(i) \,, \qquad \text{for } 0 \leq i \leq m \,.$$

As noted earlier the function ξ_j is well-defined only for effective prices p_j^t which have positive components corresponding to desired goods, or equivalently for commodity prices π with the same property. In order to extend the definition to the entire cylinder $S \times T$, let $\beta^t = \{i : \text{good } i \text{ is desired by some consumer in } \mathcal{E}(t)\}$. Put $F_{\beta^t} = \bigcup_{i \in \beta^t} F_i$, and arbitrarily define $\xi_j(\pi,r,t) \equiv R_+^{m+1}$ for $(\pi,r,t) \in F_{\beta^t}$. Expression 5.1.6 remains valid for $(\pi,r,t) \in (S \times T)\setminus F_{\beta^t}$. Since F_{β^t} depends on the preferences of all consumers in $\mathcal{E}(t)$, artificial values may override the legitimate demands of particular groups. None of the artificial values, however, can be used or even approached by the economic algorithm because of Proposition 5.1.5 and the relationship between consumer preferences and endowments assumed below.

Consumption taxes are readily computed for values of (π,r,t) in $(S \times T)\setminus F_{\beta^t}$. The $(n-m)$-vector of such taxes paid by consumer group j of economy $\mathcal{E}(t)$ is obtained by summing for each revenue system the amounts levied on different commodities, i.e.,

$$(5.1.7) \qquad \phi_j(\xi_j, \pi, r, t) = \sum_{i=0}^{m} \Phi_j^t(\cdot,i)\, \pi(i)\, \xi_j(i; \pi, r, t) \; .$$

For values of (π,r,t) in F_{β^t} it is necessary to assign artificial values to taxes to satisfy certain technical conditions. For $j = 1,\dots,M$ put $\phi_j(\xi_j, \pi, r, t) \equiv 0$ for $(\pi, r, t) \in F_{\beta^t}$. As in the case of demands, these artificial values never arise in computations.

The aggregate components which describe the consumption side of the general economic model are obtained by summing their consumer-specific counterparts over all consumer groups. The components in question include initial endowments $w(\cdot)$, the market demand correspondence $\Xi(\cdot)$, and the consumer tax function $\phi(\cdot)$. They are defined as follows:

$$(5.1.8) \qquad w(t) = \sum_{j=1}^{M} w_j(t) \; ;$$

$$(5.1.9) \qquad \Xi(\pi,r,t) = \sum_{j=1}^{M} \xi_j(\pi,r,t) \; ;$$

$$(5.1.10) \qquad \phi(\xi,\pi,r,t) = \sum_{j=1}^{M} \phi_j(\xi_j, \pi, r, t) \; , \quad \text{where } \xi = \sum_{j=1}^{M} \xi_j \; .$$

The tax function ϕ is well-defined because the representation $\xi = \sum_{j=1}^{M} \xi_j$ is unique for $(\pi,r,t) \in (S \times T) \setminus F_{\beta t}$, and all taxes are zero for $(\pi,r,t) \in F_{\beta t}$.

In order for these aggregate components to satisfy the conditions assumed in Chapter 3, certain restrictions must be placed on the parameters that characterize the consumer groups. Sufficient conditions are provided by the following four assumptions.

5.1.11. ASSUMPTION. For each consumer group $j = 1, \ldots, M$ the parameters $w_j(t)$, $\rho_j(t)$, $b_j(t)$, a_j^t, and Φ_j^t vary continuously in t.

5.1.12. ASSUMPTION. The set of goods desired by each consumer group does not change throughout the family of economies.

5.1.13. ASSUMPTION. For each $t \in [0,1]$, $\sum_{j=1}^{M} \rho_j(t) = e$.

5.1.14. ASSUMPTION. For any desired good i and index $\ell \in \{0,\ldots,n\}$ distinct from i, there exists a consumer in $\mathcal{E}(t)$ who desires good i and possesses a positive endowment (share) of good (revenue system) ℓ.

The last assumption helps insure consistent behavior of Ξ as it diverges to $+\infty$. It can be weakened somewhat, but not eliminated entirely.

The remainder of this section consists of demonstrating that $w(\cdot)$, $\Xi(\cdot)$, and $\phi(\cdot)$ satisfy the most stringent conditions imposed in Chapter 3. Applicable conditions will be checked for each component in turn.

Properties of $w(\cdot)$

3.4.1: Continuity follows from Assumption 5.1.11.

Properties of $\Xi(\cdot)$

First it is useful to note that because of Assumption 5.1.12, the set β^t does not depend on t. The superscript will henceforth be suppressed. Because of Assumption 5.1.11, Ξ is continuous as well as single-valued on $(S \times T)\backslash F_\beta$. On F_β the correspondence assumes the artificial value R_+^{m+1}.

3.1(b): Degree zero homogeneity of $\Xi(\cdot, t)$ in (π, r) results from the same property of each ξ_j, which may be verified directly from 5.1.6.

3.4.2(a): Ξ is u.s.c. on $S \times T$ because it is single-valued and continuous on $(S \times T)\backslash F_\beta$ and assumes all possible limiting values on F_β.

3.4.2(b): Each image set $\Xi(\pi, r, t)$ is either a singleton or R_+^{m+1}, and is hence convex.

3.4.2(c): Walras Law for $\pi \gg 0$ follows directly from expressions 5.1.6 and 5.1.7 (Assumption 5.1.13 is required at the end).

3.4.2(d): All components of Ξ are bounded below by zero.

3.5.2: Put $\underline{\mathcal{L}}(\alpha) = \mathcal{L}(\alpha) \cap (S \times T)\setminus F_\beta$. Since Ξ is single-valued and continuous on $(S \times T)\setminus F_\beta$, it is clearly l.s.c. and bounded on $\underline{\mathcal{L}}(\alpha)$. Furthermore, since $(S \times T)\setminus F_\beta$ contains all points (π, r, t) in $S \times T$ with $\pi \gg 0$, part (a) also holds. It only remains to show that $\underline{\mathcal{L}}(\alpha)$ is closed. Consider any sequence $\langle \pi^k, r^k, t^k \rangle$ in $\underline{\mathcal{L}}(\alpha)$ converging to a point (π, r, t) in $S \times T$. Since $\mathcal{L}(\alpha)$ is closed (by u.s.c. of Ξ), it must contain (π, r, t). Suppose $(\pi, r, t) \in \mathcal{L}(\alpha) \cap F_\beta$. At least one component of (π, r) is positive, and at least one of the components of π^k corresponding to a desired good approaches zero. By Assumption 5.1.14 there is a consumer in $\mathcal{E}(t)$ who desires the free good and has a positive endowment (or revenue share) corresponding to the positive component of (π, r). The income of this consumer is bounded away from zero for large k, so by Proposition 5.1.5 his consumption, and hence total consumption, diverges to $+\infty$ in norm along $\langle \pi^k, r^k, t^k \rangle$. This contradicts the boundedness of Ξ on $\underline{\mathcal{L}}(\alpha)$. Hence $(\pi, r, t) \in \underline{\mathcal{L}}(\alpha)$, proving that the set is closed.

3.5.3: $\mathrm{Diam}_\infty\, \Xi(\pi, r, t) = 0$ for $\pi \gg 0$ since Ξ is single-valued at these points.

The argument used to verify Condition 3.5.2 gives a precise meaning to earlier remarks concerning the inaccessibility of artificial values of Ξ. Specifically, any sequence of bona-fide values of Ξ (e.g. demand labels) diverges to $+\infty$ in norm as prices approach areas where artificial values are assigned. Since demand labels must remain bounded (c.f. Step 7 of 3.4.15 and Step 8 of 3.5.4), the artificial values are inaccessible.

<u>Properties of</u> $\phi(\cdot)$

<u>3.4.6(a)</u>: The non-negativity of ϕ follows from that of Φ_j^t and Ξ.

<u>3.4.6(b)</u>: Consider a point (ξ,π,r,t) in $\bigcup_{v\in S\times T} \Xi(v)\times\{v\}$, and let $\langle \xi^k, \pi^k, r^k, t^k\rangle$ be a sequence in this union converging to (ξ,π,r,t). If $(\pi,r,t)\in (S\times T)\setminus F_\beta$, then eventually $\langle \pi^k, r^k, t^k\rangle \subset (S\times T)\setminus F_\beta$ because F_β is closed. Hence $\phi(\xi^k, \pi^k, r^k, t^k)\to\phi(\xi,\pi,r,t)$ by the continuity of ξ_j on $(S\times T)\setminus F_\beta$ and the continuity of Φ_j^t in t for $1\le j\le M$. If $(\pi,r,t)\in F_\beta$, then since $\xi^k\to\xi$ the argument used to verify Condition 3.5.2 forces $\langle\pi^k, r^k, t^k\rangle$ to eventually lie in F_β. Hence for large k, $\phi(\xi^k, \pi^k, r^k, t^k) = 0 = \phi(\xi,\pi,r,t)$.

<u>3.4.6(c)</u>: Since at least one good is desired by every consumer, all points $(0,r,t)$ lie in F_β. Hence $\phi(\xi,0,r,t) = 0$ by definition.

<u>3.4.6(d)</u>: Since Ξ is single-valued on $(S\times T)\setminus F_\beta$, ϕ is trivially affine on $\Xi(v)\times\{v\}$ for $v\in(S\times T)\setminus F_\beta$. For $v\in F_\beta$, ϕ is identically zero on $\Xi(v)\times\{v\}$ and hence affine there, too.

All requisite properties of the market aggregates $w(\cdot)$, $\Xi(\cdot)$, and $\phi(\cdot)$ have now been verified.

5.2. Activity Analysis Production

The production model described in this section is one of the simplest possible examples of the general CRS technology postulated in Chapter 3. Like conventional linear programs, the model treats production as a finite set of activities operated simultaneously at non-negative levels. Activities can use multiple inputs and produce multiple outputs. Unlike most LP's, however, the entire technology matrix can be varied parametrically.

Only two sets of parameters are required to specify the activity analysis model. Let $t \in [0,1]$ be an economy index. Production in $\mathcal{E}(t)$ is characterized by

(a) N non-slack production activities $x_1^t, \ldots, x_N^t \in R^{m+1}$;

(b) N matrices $\Gamma_1^t, \ldots, \Gamma_N^t \in R_+^{(n-m)\times(m+1)}$ of ad valorem production tax rates.

The matrices are associated with the non-slack activities on a one-to-one basis. Each row of Γ_ℓ^t corresponds to a revenue system, and each column to a commodity. As required by the general model, free disposal is implicit in the technology of every economy.

The non-slack production activities may be grouped together into sectors or assigned to particular firms in any desired manner. Since CRS technologies generate no profits at equilibrium, ownership patterns are irrelevant to economic behavior. For expository convenience, however, the N activities available to each economy will be considered in this chapter as separate sectors.

Collectively the individual sectors define the non-slack unit *activity correspondence*

$$(5.2.1) \qquad \mathcal{B}(t) = \{x_1^t, \ldots, x_N^t\} .$$

Given a unit activity $b \in \mathcal{B}(t)$, the vector of unit tax liabilities incurred by b at prices (π, r) is

$$(5.2.2) \qquad \gamma(b,\pi,r,t) = \sum_{i=0}^{m} \Gamma_\ell^t(\cdot,i)\ \pi(i)\ |x_\ell^t(i)| ,$$

where $b = x_\ell^t$. If b resides in more than one sector, then the tax rates in the overlapping sectors are required to agree. This insures that γ is well-defined. The constraint can be eliminated, however, by the simple extension of the economic model described in Section 3.6.

To insure that $\mathcal{B}(\cdot)$ and $\gamma(\cdot)$ satisfy the conditions assumed in Chapter 3, two restrictions must be placed on the defining parameters.

5.2.3. ASSUMPTION. For $\ell = 1, \ldots, N$ the parameters x_ℓ^t and Γ_ℓ^t vary continuously in t.

5.2.4. ASSUMPTION. For each $\ell = 1, \ldots, N$ the activity x_ℓ^t uses an input which cannot be produced in any economy in a neighborhood N_ℓ^t of t.

The latter assumption prevents nearby economies from producing something out of nothing, but is clearly much stronger than necessary to achieve this effect. It has the advantage, however, of being easy to check and is broad enough to encompass the numerical examples considered in the next chapter.

The two preceding assumptions are sufficient to force $\mathcal{B}(\cdot)$ and $\gamma(\cdot)$ into the mold of the general theory. Pertinent conditions will be checked for $\mathcal{B}(\cdot)$ and $\gamma(\cdot)$ in turn.

Properties of $\mathcal{B}(\cdot)$

3.4.3: Each x_ℓ^t for $\ell = 1, \ldots, N$ is a continuous function of t on $[0,1]$, and hence a continuous bounded correspondence. Thus $\mathcal{B}(t)$, the union of these correspondences, is also continuous and bounded.

3.4.4: According to 5.2.4 each activity x_ℓ^t contains a component $x_\ell^t(i) < 0$ such that for $1 \leq j \leq N$, $x_j^{t'}(i) \leq 0$ on N_ℓ^t. Since x_ℓ^t is continuous in t, there is a sub-neighborhood of N_ℓ^t on which $x_\ell^{t'}(i) < 0$. Let N^t be the intersection of these sub-neighborhoods for $\ell = 1, \ldots, N$. Then no collection of activities drawn from the economies spanned by N^t can be operated at semi-positive levels without inputs. In view of Remark 3.4.5, Condition 3.4.4 holds.

Properties of $\gamma(\cdot)$

3.1(e): Degree one homogeneity of γ in (π, r) is an immediate consequence of γ's linearity in these variables.

5.4.7(a): Since $\Gamma_\ell^t \geq 0$, the non-negativity of γ is apparent from 5.2.2.

5.4.7(b): Consider (b,π,r,t) in $\bigcup_{t\in T} \mathcal{B}(t) \times S \times \{t\}$, and let $\langle b^k, \pi^k, r^k, t^k\rangle$ be a sequence in this union converging to (b,π,r,t). Clearly b^k must lie in some sector ℓ infinitely often. Thus $b^k = x_\ell^{t^k} \to x_\ell^t = b$ along a corresponding subsequence. Equation 5.2.2 holds along this subsequence, so the continuity of Γ_ℓ^t in t implies $\gamma(b^k, \pi^k, r^k, t^k) \to \gamma(b,\pi,r,t)$. This suffices to establish the continuity of γ on $\bigcup_{t\in T} \mathcal{B}(t) \times S \times \{t\}$.

5.4.7(c): It is obvious from 5.2.2 that $\gamma(b,0,r,t) = 0$.

All requisite properties of $\mathcal{B}(\cdot)$ and $\gamma(\cdot)$ have now been verified. Before leaving this section it is worth mentioning how labels are computed in the activity analysis model. Given a vertex $(\pi,r,t) \in S \times T$ with $(\pi,r) \gg 0$, one simply evaluates $\pi x_\ell^t - e\gamma(x_\ell^t, \pi, r, t)$ for each sector ℓ in turn, stopping when a positive value is found. If no such values are found, then a demand label is installed. Otherwise the label becomes

$$L(\pi,r,t) = \begin{bmatrix} -x_\ell^t \\ \gamma(x_\ell^t, \pi, r, t) \end{bmatrix},$$

where ℓ is the first sector earning a positive after-tax profit.

5.3. CES Production

The CES production model resembles the activity analysis model in that every economy $\mathcal{E}(t)$ admits a finite number of sectors. Each sector of $\mathcal{E}(t)$, however, has available a continuum of unit activities rather than a finite number. This permits continuous substitution among input factors as commodity prices vary. Ad valorem production taxes identical to those of the previous section are imposed on all unit activities.

The technology of each sector is determined by a CES production function of essentially the same form as the utility function employed in Section 5.1. In a typical sector the $m+1$ commodities are partitioned into a non-empty set α of inputs and a non-empty set β of outputs. Feasible production plans $x \in R^{m+1}$ have non-positive input components $x(\alpha)$ and non-negative output components $x(\beta)$. Inputs and outputs are related by

$$x(\beta) = o(\beta)\, z(c,\ d,\ x(\alpha)) \ , \tag{5.3.1}$$

where $o(\beta)$ is a fixed vector of non-negative output levels and

$$z(c,\ d,\ x(\alpha)) = \left[\sum_{i \in \alpha} c(i)\,|x(i)|^{(d-1)/d} \right]^{d/(d-1)} \tag{5.3.2}$$

is a scalar-valued CES production function. The parameters of z are confined as in Section 5.1 to values that yield concave non-decreasing functions, i.e., substitution elasticities $d > 0$ and factor weights $c \geq 0$. (The expression for z is superficially different from 5.1.1 in that the weights are unexponentiated.) For $d = 1$ the Cobb-Douglas form with $ec = 1$ is used. Since any input with $c(i) = 0$ can be reclassified as an output with $o(i) = 0$, all factor weights are assumed to be positive.

Output proportions are unaffected by changes in inputs -- only output levels vary. Moreover, the technology exhibits constant returns to scale because of the degree one homogeneity of z in $x(\alpha)$. Hence the relationship between inputs and outputs may be completely summarized by any set of "unit" activities consisting of one point along each ray connecting the origin of R^{m+1} with a feasible production plan. The particular set of unit activities selected for use here are those lying on the unit ℓ_∞-sphere. Specifically, the non-slack unit activities in a typical sector consist of the set

$$X(o(\beta),\ c,\ d) = \{x \in R^{m+1} : \|x\|_\infty = 1,\ x(\alpha) \leq 0,$$

$$\text{and}\quad x(\beta) = o(\beta)\ z(c,d,x(\alpha))\} \quad .$$

The general model presumes that all production plans in pos $X(o(\beta),c,d)$ are technically feasible. Actually only a few such plans satisfy 5.3.1. It will later be demonstrated, however, that any activities from the same sector appearing as labels will be virtually identical, provided the vertices bearing the labels are close together. Hence any production plan formed from such labels will closely resemble a true CES production plan.

The manner in which the activity set $X(o(\beta), c, d)$ moves in response to changes in the defining parameters is of interest in determining the kinds of deformations possible in the CES model. The following proposition insures that $X(\cdot)$ possesses sufficient regularity to tolerate a wide range of technological deformations.

5.3.3. PROPOSITION. The set $X(o(\beta), c, d)$ varies continuously as a correspondence in $o(\beta)$, c, and d over all values of $o(\beta) \geq 0$, $d > 0$, and $c \gg 0$ s.t. $ec = 1$.

<u>Proof</u>: Upper semi-continuity will be demonstrated first. Let $\langle o^k(\beta), c^k, d^k\rangle$ be a sequence of admissible parameters converging to the admissible triple $\langle o(\beta), c, d\rangle$, and let $x^k \to x$ where $x^k \in X(o^k(\beta), c^k, d^k)$. Clearly $\|x\|_\infty = 1$ and $x(\alpha) \leq 0$. Since $x^k(\beta) \to x(\beta)$ and $o^k(\beta) \to o(\beta)$, all that must be established is that $z(c^k, d^k, x^k(\alpha)) \to z(c, d, x(\alpha))$. For $d > 1$ Expression 5.3.2 is jointly continuous in $(c, d, x(\alpha))$, so the desired convergence takes place. For $d < 1$ the same reasoning applies when $x(\alpha) \ll 0$. If some component of $x(\alpha)$ is zero, replace that component by $\epsilon > 0$, and the resulting expression dominates $z(c^k, d^k, x^k(\alpha))$ for large k. As $k \to \infty$ the dominating expression converges in the normal manner. By shrinking $\epsilon \to 0$, $\limsup_{k\to\infty} z(c^k, d^k, x^k(\alpha))$ is forced down to zero, which is the imputed value of $z(c, d, x(\alpha))$.

For $d = 1$ separate arguments are required for subsequences of $\langle d^k\rangle$ less than, equal to, and greater than one. The case where $d^k = 1$

is trivial. The other cases may be handled by sandwiching $z(c^k, d^k, x^k(\alpha))$ between two similar expressions with constant input levels and weights. The bounding expressions converge to Cobb-Douglas forms as $k \to \infty$ via L'Hospital's rule, and the limiting sandwich collapses to the desired value. The sandwich is constructed from input levels $(|x(i)| \pm \epsilon)^+$ and factor weights $c(i) \pm \delta$, where the plus (minus) sign applies to weights on perturbed input levels above (below) the median. The condition $ec^k = 1$ makes the weight perturbation work. The detailed argument is quite tedious and hence omitted.

Lower semi-continuity is established next. As above suppose $\langle o^k(\beta), c^k, d^k\rangle$ converges to $\langle o(\beta), c, d\rangle$, and let $x \in X(o(\beta), c, d)$. Define $x^k(\alpha) = x(\alpha)$ and let $x^k(\beta) = o^k(\beta)\, z(c^k, d^k, x^k(\alpha))$. Then $x^k/\|x^k\|_\infty$ belongs to $X(o^k(\beta), c^k, d^k)$, and by the same reasoning used to establish convergence in the upper semi-continuity proof, $x^k(\beta) \to x(\beta)$. Hence $x^k \to x$, and since $\|x\|_\infty = 1$, $x^k/\|x^k\|_\infty \to x$. □

Now that the technology of a typical sector has been investigated, a comprehensive specification of the CES model is possible. Each economy $\mathcal{E}(t)$ for $t \in [0,1]$ has N production sectors indexed $\ell = 1, \ldots, N$. Each sector is characterized by the following parameters:

(a) Non-empty sets of inputs α_ℓ and outputs β_ℓ satisfying $\alpha_\ell \cup \beta_\ell = \{0, \ldots, m\}$. (Note that these sets are independent of t.)

(b) A vector $o^t_\ell(\beta_\ell) \in R_+^{\text{card}\,\beta_\ell}$ of unit output levels.

(c) A CES production function of the form 5.3.2 with substitution elasticity $d_\ell(t) > 0$ and factor weights $c^t_\ell(\alpha_\ell)$ satisfying $c^t_\ell \gg 0$ and $ec^t_\ell = 1$.

(d) A matrix $\Gamma_\ell^t \in R_+^{(n-m)\times(m+1)}$ of ad valorem tax rates. Each row corresponds to a revenue system and each column to a commodity.

In terms of these parameters, sector ℓ of economy $\mathcal{E}(t)$ possesses a non-slack unit activity set $\mathcal{B}_\ell(t) = X(o_\ell^t(\beta_\ell), c_\ell^t, d_\ell(t))$. Combining these activity sets for all sectors yields the economy-wide activity correspondence $\mathcal{B}(t) = \bigcup_{\ell=1}^{N} \mathcal{B}_\ell(t)$. As always the activities in $\mathcal{B}(t)$ are complemented by the $m+1$ free disposal activities.

Given a production plan $b \in \mathcal{B}(t)$, the vector of unit tax liabilities incurred by b at prices (π, r) is

$$\gamma(b,\pi,r,t) = \sum_{i=0}^{m} \Gamma_\ell^t(\cdot,i)\ \pi(i)\ |b(i)|\ ,$$

where $b \in \mathcal{B}_\ell(t)$. If b belongs to more than one sector, then the tax rates in the overlapping sectors are presumed to agree. As noted in the previous section this restriction can be lifted by extending the general model in the manner described in Section 3.6.

Sufficient conditions for $\mathcal{B}(\cdot)$ and $\gamma(\cdot)$ to satisfy all requirements of Chapter 3 are contained in the following two assumptions.

5.3.4. ASSUMPTION. For $\ell = 1, \ldots, N$ the parameters $o_\ell^t(\beta_\ell)$, $c_\ell^t(\alpha_\ell)$, $d_\ell(t)$, and Γ_ℓ^t vary continuously in t.

5.3.5. ASSUMPTION. For each sector ℓ and economy $\mathcal{E}(t)$ there is a neighborhood N_ℓ^t of t and a set of inputs, at least one of which is used by every activity in sector ℓ for all economies in N_ℓ^t, and which cannot be produced by any economy in N_ℓ^t.

The latter assumption, like its counterpart in the activity analysis section, is much stronger than necessary to insure technical realism, yet broad enough to cover a wide class of examples. With these assumptions in hand the activity correspondence $\mathcal{B}(\cdot)$ and tax function $\gamma(\cdot)$ are readily shown to fit the pattern of the general model.

<u>Properties of</u> $\mathcal{B}(\cdot)$

<u>3.4.3</u>: Proposition 5.3.3 in conjunction with Assumption 5.3.4 implies that $\mathcal{B}_\ell(\cdot)$ is a continuous correspondence w.r.t. t. Since the images $\mathcal{B}_\ell(t)$ lie on the unit ℓ_∞-sphere of R^{m+1}, $\mathcal{B}_\ell(\cdot)$ is also bounded. Consequently the union $\mathcal{B}(\cdot)$ is continuous and bounded.

<u>3.4.4</u>: Let $N^t = \bigcap_{\ell=1}^{N} N^t_\ell$, and consider any set of activities drawn from $\mathcal{B}(N^t)$. By Assumption 5.3.5 each activity in the set requires an input which cannot be supplied by the others. Hence no combination of the activities can be operated at positive levels in the absence of external resources. In view of Remark 3.4.5, Condition 3.4.4 holds.

<u>Properties of</u> $\gamma(\cdot)$

The tax function γ is identical in form to the one defined in the previous section and exhibits the same properties for exactly the same reasons.

The process of generating labels for the CES production model is much more complicated than the activity analysis model because of the continuum of unit activities in each sector. The object, of course, is the same: given a vertex (π, r, t) with $(\pi, r) \gg 0$, find an activity in $\mathcal{B}(t)$ with positive after-tax profits or show that none exists. The search for a profitable activity must be conducted on a sector-by-sector basis.

Consider the after-tax profitability of a unit activity x in sector ℓ of economy $\mathcal{E}(t)$. At prices $\pi \gg 0$ activity x earns

$$(5.3.6) \qquad \pi x - e\gamma(x,\pi,r,t) = \pi x - \sum_{i=0}^{m} e\Gamma_{\ell}^{t}(\cdot,i)\, \pi(i)\, |x(i)|$$

$$= \pi(\beta)\, x(\beta) - \sum_{i \in \beta} e\Gamma(\cdot,i)\, \pi(i)\, |x(i)|$$

$$+ \pi(\alpha)\, x(\alpha) - \sum_{i \in \alpha} e\Gamma(\cdot,i)\, \pi(i)\, |x(i)| \,,$$

where the subscript ℓ and superscript t have been suppressed in the last expression (and will continue to be). The sign of 5.3.6 is clearly the same for all positive multiples of x. Suppose the substitution elasticity $d > 1$. Then $\|x\|_{\infty} = 1 \Rightarrow x \neq 0 \Rightarrow x(\alpha) \neq 0 \Rightarrow z(c,d,x(\alpha)) > 0 \Rightarrow$ a positive multiple $\bar{x}$ of x exists such that $z(c, d, \bar{x}(\alpha)) = 1$. Hence the search for a profitable activity in this sector may equally well take place on $X' = \{x \in R^{m+1} : x(\beta) = o(\beta),\ x(\alpha) \leq 0, \text{ and } z(c,d,x(\alpha)) = 1\}$.

The same conclusion holds for $d \leq 1$, but for slightly different reasons. Since $z(c, d, x(\alpha)) > 0$ for $x(\alpha) \ll 0$, points in $X(o(\beta),c,d)$ with $x(\alpha) \ll 0$ may be scaled to lie in X'. If, however, $x(i) = 0$ for some $i \in \alpha$, then $x(\beta) = 0$, and it is clear from 5.3.6 that the profitability of x is non-positive. Hence such points may be excluded from the search.

The problem of finding a profitable activity in the set X' is much easier than finding one in $X(o(\beta), c, d)$. In fact the most profitable activity in X' is easily calculated. Since outputs are constant on X', the problem reduces to finding the input mix which minimizes factor costs and taxes, i.e.,

$$\text{(5.3.7)} \qquad \begin{aligned} &\text{minimize} && p(\alpha)\, x(\alpha) \\ &\text{subject to} && z(c, d, x(\alpha)) = 1 \\ &&& x(\alpha) \geq 0 , \end{aligned}$$

where the $x(\alpha)$ are absolute input levels and $p(i) = (1 + e\Gamma(\cdot,i))\, \pi(i)$ for $i \in \alpha$ are effective prices (including tax) paid for the inputs. Since z increases with $x(\alpha)$ and since inputs cost money, the equality constraint in 5.3.7 can be relaxed to $z(c, d, x(\alpha)) \geq 1$ without affecting the solution. The result is a nice convex program, differentiable for $x(\alpha) \gg 0$. Analyzing the Kuhn-Tucker sufficient conditions for this program yields a unique cost minimizing solution

$$\text{(5.3.8)} \qquad \bar{x}(i) = \left[\frac{c(i)}{p(i)} \right]^{d} \left[\sum_{i \in \alpha} c(i)^{d}\, p(i)^{1-d} \right]^{d/(1-d)}$$

for $i \in \alpha$ and $d \neq 1$. If $d = 1$ the limiting form

$$\text{(5.3.9)} \qquad \bar{x}(i) = \frac{c(i)}{p(i)} \prod_{i \in \alpha} \left[\frac{p(i)}{c(i)} \right]^{c(i)}$$

must be used.

The profit maximizing activity in the set X' is, therefore,

$$x^* = (x^*(\beta),\ x^*(\alpha)) = (o(\beta),\ -\bar{x}(\alpha)) .$$

If x^* shows a positive after-tax profit, as measured by 5.3.6, then the rescaled unit activity $x^*/\|x^*\|_\infty \in X(o(\beta),\ c,\ d)$ together with its taxes may be taken as the production label. If, however, x^* earns a non-positive profit, then so does every unit activity in $X(o(\beta),\ c,\ d)$, and the current sector must be bypassed in the label search. As was the case in the activity analysis model, the search proceeds sector-by-sector until a positive profit is found or until the sectors are exhausted.

The discrepancy noted earlier between the true CES production set $\{\lambda \mathcal{B}_\ell(t) : \lambda \geq 0\}$ for sector ℓ of economy $\mathcal{E}(t)$ and the set pos $\mathcal{B}_\ell(t)$ assumed in the general model disappears for all practical purposes due to the nature of the labeling. One can show through arguments similar to those used to establish Proposition 5.3.3 that the optimal input mix $\bar{x}(\alpha)$ given in 5.3.8 and 5.3.9 varies continuously in (π, r, t) provided $\pi \gg 0$. (Recall that the t and ℓ-parameters are suppressed in these expressions.) Consequently the label candidate $x^*/\|x^*\|_\infty$ is also continuous in (π, r, t). Whenever two or more such candidates are used to label an n-simplex encountered by the economic algorithm, then provided the n-simplex has a small diameter (as is customary along approximate equilibrium graphs), the labels will be virtually identical. Hence any pseudo-production plan determined by these labels will approximate some member of $\{\lambda \mathcal{B}_\ell(t') : \lambda \geq 0\}$ for all economies $\mathcal{E}(t')$ near $\mathcal{E}(t)$.

CHAPTER 6

COMPUTATIONAL EXPERIENCE

The thrust of the preceding chapters has been to develop the theory of a computational algorithm which, when implemented on a computer, can generate explicit numerical approximations to equilibrium graphs. The purpose of the present chapter is to describe the outcome of a series of numerical experiments in which programs implementing the algorithm were used to compute approximate equilibrium graphs for explicitly deformed economies. The results of these experiments are presented from two perspectives. First, operating statistics such as precision of approximation and measures of computational effort are tabulated and analyzed. Second, portions of selected equilibrium graphs are displayed and interpreted. In keeping with the methodological orientation of this study, primary emphasis is placed on the former perspective.

The computer programs were applied to approximately twenty test problems, of which thirteen are reported here. (The others involved single economy runs or runs with the identity deformation.) All thirteen problems are special cases of the production and consumption models presented in Chapter 5. Although limited in scope relative to Chapter 5's possibilities, these examples nevertheless typify the empirical models currently (1975) in use. Among them can be found deformations of every major component of the general economic model except consumer taxes. Some of the problems reconstruct John Shoven's analysis of the effects of differential capital income taxation in the U.S. The results of these experiments confirm that the pairs of equilibria compared by Shoven are connected by equilibrium graphs spanning the evenly and unevenly taxed economies.

The overriding message that emerges from the numerical experiments is that approximating equilibrium graphs with precision involves a tremendous amount of computational effort. Tens and perhaps hundreds of thousands of iterations are required to solve models of even modest size. The final section of this chapter identifies the major factors responsible for this unwonted behavior, and concludes that such computations are inherently expensive because of the vast amounts of information represented by densely defined equilibrium paths.

6.1. Description of Test Problems

The thirteen test problems were derived from three basic economic models by subjecting each to various deformations. Many of the problems exclude the revenue systems of the basic model and hence fall under the first variation of the general theory described in Section 3.6. Apart from the absence of revenue systems, each test problem fits two of the three formulations presented in Chapter 5, and will hence be specified in terms of the relevant Chapter 5 parameters.

The deformations in all test problems result from linear changes in the defining parameters. Hence each family of economies $\{\mathcal{E}(t)\}_{t\in[0,1]}$ is completely specified once the parameter values for $\mathcal{E}(0)$ and $\mathcal{E}(1)$ are known. Since many parameters have the same value at both endpoints, the most efficient procedure is to display all parameters for $\mathcal{E}(0)$ and then identify the ones that change during the deformation.

The first basic economic model is the six-commodity hypothetical economy appearing in Section 5.3 of Scarf's monograph [13]. The model features activity analysis production in a single time period setting. No revenue systems are included. The commodities traded in the model have the following interpretations:

Commodity	Description
0	Capital available at end of period
1	Capital available at beginning of period
2	Skilled labor
3	Unskilled labor
4	Nondurable consumer goods
5	Durable consumer goods

Five groups of consumers participate in this hypothetical economy. Their behavior for $\varepsilon(0)$ is characterized by the parameters displayed in Tables 6.1.1, 6.1.2, and 6.1.3. No revenue share factors or consumer tax rates are needed since no revenue systems appear in the model.

Initial holdings exist for all commodities except non-durable consumer goods and end-of-period capital. Each consumer is endowed with the potential to provide a fixed number of hours of labor services. Any portion of the labor endowment, however, may be consumed as leisure. According to conventional measures of wealth, consumer group 5 with 6.0 units of beginning capital is the richest, while group 2 with 0.1 units is the poorest. (These holdings will be reversed in one of the test

TABLE 6.1.1. Initial Endowments $w_j(0)$

Commodity	Consumer Group 1	2	3	4	5
0	0.0	0.0	0.0	0.0	0.0
1	3.0	0.1	2.0	1.0	6.0
2	5.0	0.1	6.0	0.1	0.1
3	0.1	7.0	0.1	8.0	0.5
4	0.0	0.0	0.0	0.0	0.0
5	1.0	2.0	1.5	1.0	2.0

TABLE 6.1.2. Demand Intensities a_j^0

Commodity	Consumer Group 1	2	3	4	5
0	4.0	0.4	2.0	5.0	3.0
1	0.0	0.0	0.0	0.0	0.0
2	0.2	0.0	0.5	0.0	0.0
3	0.0	0.6	0.0	0.2	0.2
4	2.0	4.0	2.0	5.0	4.0
5	3.2	1.0	1.5	4.5	2.0

TABLE 6.1.3. Substitution Elasticities $b_j(0)$

	Consumer Group			
1	2	3	4	5
1.2	1.6	0.8	0.5	0.6

problems.) All goods except beginning capital are desired by at least one consumer.

The production side of the economy is represented by an activity analysis matrix consisting of eight non-slack unit activities. The $\mathcal{E}(0)$ coefficients for these activities are shown in Table 6.1.4. No producer taxes are included in this formulation. The activities are

TABLE 6.1.4. Unit Production Activities x_ℓ^0

		Non-Slack Activity							
		1	2	3	4	5	6	7	8
Commodity	0	4.0	4.0	1.6	1.6	1.6	0.9	7.0	8.0
	1	-5.3	-5.0	-2.0	-2.0	-2.0	-1.0	-4.0	-5.0
	2	-2.0	-1.0	-2.0	-4.0	-1.0	0.0	-3.0	-2.0
	3	-1.0	-6.0	-3.0	-1.0	-8.0	0.0	-1.0	-8.0
	4	0.0	0.0	6.0	8.0	7.0	0.0	0.0	0.0
	5	4.0	3.5	0.0	0.0	0.0	0.0	0.0	0.0

grouped into three sectors: a durable goods sector consisting of activities 1 and 2; a non-durable goods sector consisting of activities 3, 4, and 5; and a capital formation sector consisting of activities 6, 7, and 8. All sectors use varying amounts of labor and beginning capital as inputs, and generate ending capital as a by-product.

Starting with the above values for $\mathcal{E}(0)$, linear deformations were applied to selected parameters to yield three test problems. The deformations alter resource ownership, consumer tastes, and production technology. Parameters affected by the deformations are noted below for each problem.

Test Problem 1

The initial endowments of beginning capital held by the "richest" and "poorest" consumers are reversed. Thus in $\mathcal{E}(1)$ consumer group 2 holds 6.0 units of beginning capital while consumer group 5 holds 0.1 units.

Test Problem 2

The substitution elasticities of all consumer groups are deformed to the Cobb-Douglas value of 1.0.

Test Problem 3

The productivity of labor in all sectors is doubled. This is accomplished by cutting the labor input coefficients in half, i.e., for $\ell = 1, \ldots, 8$ and $i = 2, 3,$ $x_\ell^1(i) = \frac{1}{2} x_\ell^0(i)$.

The three test problems satisfy all the applicable conditions of Chapter 5 except 5.1.14. This condition is violated because no consumer owns an initial endowment of ending capital or non-durable goods. The breach of this condition, however, had no discernable effect on the performance of the algorithm because all prices along the equilibrium graphs were positive. (The condition serves only to prevent pathological behavior of demand as the prices of desired goods approach zero.) If difficulties had arisen, the benefits of Condition 5.1.14 could have easily been secured by a negligible adjustment to initial endowments, such as assigning holdings of 0.000001 in goods 0 and 4 to each consumer group.

The second and third basic economic models are, respectively, the four and fourteen commodity empirical models of the U.S. economy used by John Shoven (and John Whalley) to estimate the efficiency loss induced by unequal rates of taxation on the income from capital employed in different economic sectors [16], [17], [18]. Both models are based on the same empirical data. They differ only in the number of sectors into which the data are aggregated (two versus twelve). CES production functions are used to describe the technology in each case.

Both models incorporate a single revenue system. Prices and revenue levels are scaled, however, to lie on the transformed simplex $S' = \{(\pi, r) \in R_+^{n+1} : e\pi + 0.026905r = 1\}$ rather than the standard simplex S. One can easily verify that the theory of Chapters 2, 3, and 5 remains intact with S' replacing S. As for manifolds on $S' \times [0,\infty)$, one need only multiply the r-component of each abstract vertex in $S \times [0,\infty)$ by the reciprocal of 0.026905 to obtain an equivalent vertex in $S' \times [0,\infty)$.

Owing to the scarcity of empirical data, some of the parameters in Shoven's models had to be estimated exogenously. Such parameters were typically assigned a range of likely values, each resulting in a separate version of the models. Of the many cases considered by Shoven, five were selected for analysis here.

The detailed specification begins with the four-commodity basic economy and its attendant test problems. The goods comprising this model may be described as follows:

Commodity	Description
0	"Non-corporate" outputs
1	"Corporate" outputs
2	Labor
3	Capital

The terms "corporate" and "non-corporate" are merely suggestive; a more meaningful description of these categories will be provided when the disaggregated model is introduced.

U.S. consumers are divided into two large groups, one representing the upper ten percent of all income recipients and the other the lower ninety percent. The behavior of these groups for $\mathcal{E}(0)$ is determined by the parameters contained in Tables 6.1.5 through 6.1.8. Consumer tax rates Φ_j^0 for $j = 1, 2$ are not shown in the tables because they are identically zero.

TABLE 6.1.5. Initial Endowments $w_j(0)$

Commodity	Consumer Group 1	Consumer Group 2
0	0.0	0.0
1	0.0	0.0
2	49.3959	167.9461
3	16.8416	25.2624

TABLE 6.1.6. Revenue Shares $\rho_j(0)$

Consumer Group 1	Consumer Group 2
0.4	0.6

TABLE 6.1.7. Demand Intensities a_j^0

Commodity	Consumer Group 1	Consumer Group 2
0	0.125	0.158744
1	0.875	0.841256
2	0.0	0.0
3	0.0	0.0

TABLE 6.1.8. Substitution Elasticities $b_j(0)$

		Consumer Group 1	Consumer Group 2
Version	C1	0.5	0.5
	C2	1.0	1.0

Initial holdings exist only for the input factors labor and capital. Finished goods are the only ones desired. Consumers, therefore, sell their entire endowments to producers and forego any labor-leisure choices. Factor supplies are thus effectively fixed. Consumer substitution elasticities had to be estimated exogenously, resulting in the two versions shown in Table 6.1.8. These will be combined with different versions of the production parameters to synthesize test problems.

Production in the four-commodity economy proceeds according to the CES model of Section 5.3. Two production sectors are involved. Each uses the two inputs labor and capital, so $\alpha_1 = \alpha_2 = \{2,3\}$. The output sets for the two sectors nominally consist of goods 0 and 1. Each sector, however, produces only one item, so the unit output vectors $o^t_\ell(\beta_\ell)$ are actually multiples of the unit vectors $e_1, e_2 \in R^2$. Factor substitution elasticities had to be estimated exogenously and were then used to derive the other parameters. Consequently Tables 6.1.9, 6.1.10, and 6.1.11 each contain three versions of the CES production parameters for $\mathcal{E}(0)$.

TABLE 6.1.9. Non-zero Components of Unit Output Vectors $o_\ell^0(\beta_\ell)$

Sector	Commodity Produced	Version P1	Version P2	Version P3
1	0	2.45070	2.44778	2.44778
2	1	1.96723	1.76534	1.54519

TABLE 6.1.10. Input Weighting Factors $c_\ell^0(\alpha_\ell)$

	Version P1		Version P2		Version P3	
Sector	Good 2	Good 3	Good 2	Good 3	Good 2	Good 3
1	0.39394	0.60606	0.35323	0.64677	0.35323	0.64677
2	0.79231	0.20769	0.88607	0.11393	0.96999	0.03001

TABLE 6.1.11. Substitution Elasticities $d_\ell(0)$

Sector	Version P1	Version P2	Version P3
1	0.99999	0.25	0.25
2	0.99999	0.75	0.50

The presence of a single revenue system in the model means that the matrix Γ_ℓ^t of production tax rates for sector ℓ collapses to a vector. All components of this vector are zero except the component corresponding to capital inputs. The non-zero component approximates the combined burden of corporate income, local property, and personal income and capital gains taxes borne by capital employed in sector ℓ. The burden is expressed as a fraction of net income received by consumers from the sale of their capital endowments to sector ℓ. The tax rates were derived independently of the substitution elasticity estimates and are hence the same for all three versions of the production parameters. Different rates are assigned to $\mathcal{E}(0)$ and $\mathcal{E}(1)$, however, reflecting the empirically observed (unequal) values and a set of hypothetical equalized values. Both sets of rates are displayed in Table 6.1.12.

TABLE 6.1.12. Non-zero Components of Production Tax Vectors Γ_ℓ^0 and Γ_ℓ^1

Sector	Commodity	Economy 0	Economy 1
1	3	0.45169	0.45169
2	3	1.22112	0.45169

A total of seven test problems were derived from the four-commodity models. Four of these suppress the revenue system and hence conform to the first variation of the general theory described in Section 3.6. Different versions of consumption and production parameters are used in the seven problems. Sometimes the parameter sets are switched during the course of the deformation. Other times the deformation involves more fundamental changes in the economies.

Test Problem 4 (revenue system excluded)

Consumption parameters C1 and production parameters P1 determine $\mathcal{E}(0)$. The deformation consists of replacing the C1 parameters by the C2 set (c.f. test problem 2).

Test Problem 5 (revenue system excluded)

The initial economy is the same as in Problem 4. Under deformation both the production and consumption parameters are replaced by the C2 and P3 versions, respectively.

Test Problem 6 (revenue system excluded)

The initial economy is characterized by consumption parameters C1 and production parameters P2. The deformation interchanges the capital endowments of the two consumer groups (c.f. test problem 1).

Test Problem 7 (revenue system excluded)

The initial economy is the same as in Problem 6. The deformation depletes the capital endowments of both consumer groups by one-half.

Test Problem 8

The economic parameters and deformation are the same as in Problem 5. This time, however, the revenue system is included, but with zero values for tax rates and revenue shares.

Test Problem 9

Consumption parameters C1 and production parameters P3 are in effect for all economies. The deformation consists of reducing capital taxes from the $\mathcal{E}(0)$ to the $\mathcal{E}(1)$ values.

Test Problem 10

This problem is identical to the preceding one except that the P2 production parameters are used throughout.

One can readily verify that the seven test problems described above satisfy all the requirements of Sections 5.1 and 5.3, except Condition 5.1.14. As was the case in the first series of test problems, this violation caused no practical problems for the algorithm, and could have easily been fixed if it did.

The fourteen-commodity basic economic model is quite similar to the four-commodity one, the only difference being the disaggregation of "corporate" and "non-corporate" outputs into a total of twelve components. Labor and capital are still the only factors of production. The full list of commodities comprising the larger model is given below. Commodities 0 through 2 were previously aggregated into "non-corporate" outputs, while goods 3 through 11 constituted the "corporate" sector.

The same consumer groups participate in the disaggregated economy as in the aggregated one. Consequently many of the same parameters apply. Revenue shares, substitution elasticities (both versions), and consumer tax rates (all zero) are identical in the two models. Initial endowments

Commodity	Description
0	Agricultural products
1	Real estate
2	Crude oil and gas
3	Minerals (other than oil and gas)
4	Contract construction
5	Manufactured goods (other than 6 and 7)
6	Lumber and wood products
7	Petroleum and coal products
8	Trade
9	Transportation
10	Communication and public utilities
11	Services
12	Labor
13	Capital

differ only in the indices of commodities. Demand intensities, however, are more numerous in the larger model. Tables 6.1.13 and 6.1.14 (the counterparts of Tables 6.1.5 and 6.1.7) contain all consumer parameters peculiar to the fourteen-commodity model.

The production side of the disaggregated model also resembles that of the aggregated one. The main difference is that twelve rather than two sectors are involved. Each sector uses labor and capital as inputs, so $\alpha_1 = \cdots = \alpha_{12} = \{12, 13\}$. The remaining commodities are nominally

TABLE 6.1.13. Initial Endowments $w_j(0)$

		Consumer Group 1	Consumer Group 2
	0-11	0.0	0.0
Commodity	12	49.3959	167.9461
	13	16.8416	25.2624

TABLE 6.1.14. Demand Intensities a_j^0

		Consumer Group 1	Consumer Group 2
	0	0.045889	0.058277
	1	0.071304	0.090553
	2	0.007807	0.009915
	3	0.011155	0.010725
	4	0.061966	0.059576
	5	0.361741	0.347791
Commodity	6	0.010905	0.010485
	7	0.016906	0.016254
	8	0.188992	0.181704
	9	0.058137	0.055893
	10	0.048154	0.046297
	11	0.117044	0.112530
	12	0.0	0.0
	13	0.0	0.0

classified as outputs for all sectors, although only one is actually produced in each. The full contingent of production parameters for the larger model appears in Tables 6.1.15 through 6.1.17. The two versions shown there correspond to versions P1 and P3 in the smaller model. These tables, together with the tax rate Table 6.1.18, constitute the fourteen-commodity counterparts of Tables 6.1.9 through 6.1.12.

TABLE 6.1.15. Non-Zero Components of Unit Output Vectors $o_\ell^0(\beta_\ell)$

Sector	Commodity Produced	Version P1	Version P3
1	0	2.34529	2.14890
2	1	2.38220	2.20603
3	2	2.06809	1.74985
4	3	1.99859	1.56676
5	4	1.33739	1.14552
6	5	2.15266	1.65283
7	6	1.94499	1.59951
8	7	2.67648	2.67633
9	8	1.86323	1.51114
10	9	1.77489	1.39553
11	10	3.08310	2.70408
12	11	1.30459	1.12974

TABLE 6.1.16. Input Weighting Factors $c_{\ell}^{0}(\alpha_{\ell})$

	Version P1		Version P3	
Sector	Good 12	Good 13	Good 12	Good 13
1	0.54051	0.45949	0.84694	0.15306
2	0.27152	0.72848	0.05835	0.94165
3	0.65054	0.34946	0.95962	0.04038
4	0.78607	0.21393	0.96811	0.03189
5	0.93311	0.06689	0.99739	0.00261
6	0.76350	0.23650	0.96349	0.03651
7	0.77163	0.22837	0.95236	0.04764
8	0.37874	0.62126	0.38391	0.61609
9	0.80001	0.19999	0.96715	0.03285
10	0.83993	0.16007	0.98451	0.01549
11	0.53260	0.46740	0.76703	0.23297
12	0.93981	0.06019	0.99784	0.00216

TABLE 6.1.17. Substitution Elasticities $d_{\ell}(0)$

Sector	Version P1	Version P3
1	0.99999	0.25
2	0.99999	0.25
3	0.99999	0.25
4	0.99999	0.50
5	0.99999	0.50
6	0.99999	0.50
7	0.99999	0.50
8	0.99999	0.50
9	0.99999	0.50
10	0.99999	0.50
11	0.99999	0.50
12	0.99999	0.50

TABLE 6.1.18. Non-Zero Components of Production Tax Vectors Γ_{ℓ}^{0} and Γ_{ℓ}^{1}

Sector	Commodity	Economy 0	Economy 1
1	13	0.42441	0.42441
2	13	0.47526	0.42441
3	13	0.25551	0.42441
4	13	1.24837	0.42441
5	13	0.96546	0.42441
6	13	1.53182	0.42441
7	13	0.75122	0.42441
8	13	0.67663	0.42441
9	13	0.83978	0.42441
10	13	1.30895	0.42441
11	13	1.53576	0.42441
12	13	0.89282	0.42441

The last three of the thirteen test problems are based on the fourteen-commodity model. As in the case of the smaller model, different combinations of consumption and production parameters are used to define the economies. Each of these problems duplicates one of the four-commodity examples in terms of the parameter sets employed and the deformation applied. Consequently each will be specified via reference to its predecessor.

Test Problem 11 (revenue system excluded)

Refer to Test Problem 4.

Test Problem 12 (revenue system excluded)

Refer to Test Problem 5.

Test Problem 13

Refer to Test Problem 9.

A summary of the distinguishing characteristics of the thirteen test problems appears in Table 6.1.19.

6.2. Behavior of Algorithm on Test Problems

The thirteen test problems were solved using a group of computer programs fashioned after the outline presented in Section 4.4. Different programs handled the two types of production and the examples with and without revenue systems. The programs were written in IBM's version of FORTRAN IV and compiled using the H-level compiler with optimization option 2 (to minimize execution time). The longest program contains more than 2000 FORTRAN source statements.

The machines used to run the programs were the IBM 360/91 and the two IBM 370/168's located at the Stanford Linear Accelerator Center. These are among the most powerful central processing units commercially available, and their power was fully utilized by the larger test problems. All but one of the problems was allocated four minutes of CPU time, yet

TABLE 6.1.19. Summary of Test Problem Specifications

Test Problem	# Goods	# Revenue Systems	Production Type	# Activities or Sectors	# Consumers	Initial Economy	Deformation
1	6	0	AA	8	5	Scarf Textbook	Exchange capital holdings of "rich" and "poor" consumers.
2	6	0	AA	8	5	Scarf Textbook	Make consumer substitution elasticities all unity.
3	6	0	AA	8	5	Scarf Textbook	Double labor productivity in all sectors.
4	4	0	CES	2	2	C1-P1	Make consumer substitution elasticities all unity, i.e., C2-P1 version.
5	4	0	CES	2	2	C1-P1	Replace parameters by C2-P3 versions.
6	4	0	CES	2	2	C1-P2	Exchange capital holdings of "rich" and "poor" consumers.
7	4	0	CES	2	2	C1-P2	Deplete capital stock by one-half.
8	4	1	CES	2	2	C1-P1	Replace parameters by C2-P3 versions.
9	4	1	CES	2	2	C1-P3	Equalize tax rates in all sectors.
10	4	1	CES	2	2	C1-P2	Equalize tax rates in all sectors.
11	14	0	CES	12	2	C1-P1	Make consumer substitution elasticities all unity, i.e., C2-P1 version.
12	14	0	CES	12	2	C1-P1	Replace parameters by C2-P3 versions.
13	14	1	CES	12	2	C1-P3	Equalize tax rates in all sectors.

this proved insufficient for two of them to run to completion (i.e., reach $\mathcal{E}(1)$). The unrestricted problem consumed 21 minutes of CPU time.

Various operating parameters had to be set for each run of the programs. These were typically adjusted by trial and error until acceptable values were found. The basis of the linear inequality system was re-inverted every 20 or 30 iterations to maintain accuracy. This proved adequate even though some columns of the basis (representing CES production activities) differed only in the seventh decimal place of a single component. Full equilibrium reports were produced each time the economy index changed by 0.1. Consequently eleven snapshots (minimum) were taken of the equilibrium graph during each complete run.

The most sensitive operating parameter turned out to be the economy index scaling factor. This constant converts vertical movements in the cylinder $S \times [0,\infty)$ (above the threshold level 2I) into changes in the economy index. Equivalently, it alters the vertical spacing of grid points. Hence it provides a means of balancing incremental shifts in economic behavior due to deformation against those caused by price changes. For large values of the scaling factor (above 1000), the algorithm floundered about the $\mathcal{E}(0)$ level, apparently unable to digest the relatively large deviations inherent in a vertical step. An exception to this behavior was observed in the examples (not reported here) involving the identity deformation. In these problems the algorithm proceeded directly to $\mathcal{E}(1)$. Like conventional equilibrium and fixed point problems, however, the labels in these examples do not change as a function of height.

For small values of the scaling factor (less than 20), the algorithm proceeded smoothly up the cylinder. Its rate of progress through the economies, however, was so slow that it seldom got beyond $\mathcal{E}(0.1)$ before the allotted time expired. The optimum value appeared to lie in the range of 100 to 150. The smaller values yielded better performance in problems involving relatively "severe" deformations, and conversely. Scaling factors near the optimum value were used in the runs summarized below.

Another sensitive set of operating parameters were the range error tolerances described in Section 4.3. Virtually all the components of range error observed in the equilibrium reports for the thirteen test problems satisfied the loose tolerances. Roughly half satisfied the central tolerances, while very few satisfied the tight tolerances. Often a single critical commodity appeared to determine the outer limit of grid size. These findings indicate that the dynamic control mechanism was successful in maintaining relatively uniform levels of range error along the approximate equilibrium graphs.

On the whole the market tolerances tended to be harder to satisfy than the profitability tolerances. Cutting market tolerances in half frequently reduced profitability errors by an order of magnitude. Tighter tolerances of either type resulted in finer grids and hence more iterations. The tolerances used in the runs reported below represent a compromise between accuracy and computational expense.

Detailed statistics from the best runs of the thirteen test problems are displayed in Table 6.2.1. A quick glance at this table reveals several

TABLE 6.2.1. Computational Statistics

Test Problem	# Goods + Revenue Systems	Profit Tolerances -Loose -Central -Tight	Market Tolerances (%) -Loose -Central -Tight	Iterations At Economy 0	Iterations At Final Economy	Index Of Final Economy	Grid Size ($\times 10^{-7}$) -Maximum -Average* -Minimum	# Manifold Blocks	CPU-Time (Minutes)	CPU Type
1	6	0.0005 0.0001 0.00001	0.5 0.1 0.01	422	64,431	1.01	153 150 76	4	2.66	360/91
2	6	0.0005 0.0001 0.00001	0.5 0.1 0.01	422	17,472	1.01	153 130 76	4	0.86	360/91
3	6	0.0005 0.0001 0.00001	0.5 0.1 0.01	422	161,009	0.97	610 200 76	50	4.00	370/168
4	4	0.001 0.0005 0.0001	1.0 0.5 0.1	201	4,471	1.01	610 200 76	84	0.12	370/168
5	4	0.001 0.0005 0.0001	1.0 0.5 0.1	201	21,740	1.03	610 250 153	94	0.54	370/168
6	4	0.001 0.0005 0.0001	1.0 0.5 0.1	213	4,453	1.02	610 230 76	41	0.12	370/168
7	4	0.001 0.0005 0.0001	1.0 0.5 0.1	213	58,487	1.01	305 170 76	169	1.55	370/168
8	5	0.001 0.0005 0.0001	5.0 1.0 0.3	170	11,536	1.06	1,221 870 610	10	0.28	370/168
9	5	0.001 0.0005 0.0001	5.0 1.0 0.3	308	15,536	1.01	2,441 1,000 76	16	0.43	370/168
10	5	0.001 0.0005 0.0001	5.0 1.0 0.3	311	15,540	1.11	2,441 1,200 610	16	0.42	370/168
11	14	0.001 0.0005 0.0001	2.0 1.0 0.5	4,823	24,403	1.03	610 350 153	54	2.23	360/91
12	14	0.001 0.0005 0.0001	2.0 1.0 0.5	4,823	42,602	0.30	1,221 500 305	20	4.00	360/91
13	15	0.001 0.0005 0.0001	5.0 1.0 0.3	13,768	206,355	1.08	1,221 890 305	6	21.33	360/91

*Approximate.

numbers that are strikingly disproportionate to those normally encountered in equilibrium and fixed point calculations. Most notable are the total numbers of iterations required to solve the problems. These range from a few thousand to hundreds of thousands, with the majority lying in the tens of thousands range. The number of iterations required to reach an approximate equilibrium for $\mathcal{E}(0)$ account for an insignificant fraction of the total. Yet these seemingly negligible values are the ones that should be compared with the results of traditional problems of this size.

The principal explanation for the inordinately large numbers of iterations can be found in the column entitled "Grid Size". The manifold mesh required to satisfy the specified tolerances ranged from ten-thousandths of a unit down to millionths, with typical values in hundred-thousandths. Such tiny simplices are rarely encountered in equilibrium or fixed point calculations, outside of Newtonian termination or acceleration routines. Yet the economic algorithm had to pivot through regions covered by simplices of this size. Homotopy-type fixed point algorithms reach such diameters only at the 15th or 16th level of refinement, and then they stop. At this point the economic algorithm is just getting started. A reasonable analogy would be to operate Scarf's algorithm with 50,000 grid points along each edge of the simplex instead of the usual hundred or so. In the next section the connection between grid size and number of iterations will be made more precise.

Although the total iteration counts for the test problems are large as a group, sizeable differences exist from problem to problem. Few of these differences can be attributed to grid size. On the contrary iteration counts appear to be negatively correlated with average grid size. Some of the differences can be explained, however, by the dimensionality of the problems. The 14 and 15-"commodity" runs, for example, tended to require many more iterations than the 4,5, and 6-"commodity" runs (problem 12 must be adjusted for the fact that it only reached $\varepsilon(0.3)$). Differences among the smaller problems cannot, however, be attributed to their dimensionality. A better explanation is afforded by the shape of the equilibrium graphs, some of which are displayed later in this section. Problems whose graphs embody the most radical price changes tended to require the most iterations. Problems with comparatively minor price variations required the fewest iterations. These facts suggest that computational effort in parametric equilibrium problems depends heavily on the severity of the applied deformation. Additional evidence in support of this hypothesis will be offered in the next section.

Despite the unprecedented numbers of iterations required to compute full equilibrium graphs, the performance of the algorithm in locating approximate equilibria for the initial economies was quite respectable. Compared with Scarf's algorithm, which is the standard technique for approximating general economic equilibria, the economic algorithm performed extremely well. Based on three single-economy examples with activity analysis production (the 6 and 14-commodity examples in Scarf's book [13] and the 7-commodity example in Shoven's dissertation [14]), the economic algorithm achieved a seven-to-tenfold improvement in iterations required to attain a given level of accuracy. Such results are consistent with similar experience in conventional fixed point problems.

Two columns of statistics in Table 6.2.1 shed some light on the performance of the dynamic control mechanism. The manifold block counts and grid size ranges indicate that substantial fluctuations in range error were detected during the course of the algorithm, and that the dynamic construction mechanism adjusted the manifold in response to these fluctuations. The observed stability of range error confirms that the adjustments had the intended effect. The most extreme variation in grid size occurred in problem 9 where six levels of refinement were used. The least variation occurred in problems 1, 2, and 8 where only two levels were used. No discernable characteristics of the test problems adequately explain the observed differences in block counts or grid size variation.

The CPU time consumed by the test problems may be analyzed in terms of CPU time per iteration times number of iterations. The time per iteration depended to a large extent on the number of dimensions in the problem, the dependence being approximately quadratic over the experimental range. Variations in the number of iterations have already been discussed.

Another perspective on time consumption is provided by Table 6.2.2. It is evident from the sample of problems presented in this table that the bulk of the processing time was expended on label generation. Production labels and demand labels on average comprised roughly equal portions of the total, although the shares in individual cases were influenced by the relative numbers of consumers and production sectors. The second most expensive activity was label system manipulation, which used about a third as much time as label generation. Basis re-inversion

TABLE 6.2.2. Percent of CPU-Time Used by Major Processing Activities

	Test Problem										
	1	2	4	5	6	7	8	9	10	11	Average
Input and Initialization	0.1	0.3	2.3	0.1	1.4	0.2	0.3	0.2	0.2	0.1	0.5
Manifold Pivot	8.6	3.0	4.2	9.8	4.5	4.7	10.0	9.5	9.2	1.3	6.5
Label Generation	56.6	67.2	58.4	56.4	57.9	64.2	47.6	52.7	51.0	61.8	57.4
Production portion	9.8	5.6	24.5	26.0	22.1	22.5	20.4	23.9	26.3	51.6	23.3
Demand portion	46.8	61.6	33.9	30.3	35.8	41.7	25.2	28.7	24.7	10.1	33.9
Label System Pivot and Basis Re-Inversion	20.1	16.4	15.9	19.7	18.9	17.9	23.5	20.7	23.2	25.9	20.2
Tolerance Checking	14.2	11.5	13.4	12.9	10.4	11.5	16.8	15.8	15.0	9.9	13.1
Equilibrium Report	0.4	1.6	5.8	1.1	6.9	1.5	1.8	1.1	1.4	1.0	2.3
TOTAL	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0

accounted for a minor fraction of this total. Surprisingly, the most logically intricate activity, manifold construction and pivoting, consumed comparatively little time.

Probably the least exciting aspect of the numerical experiments was the type of approximate equilibrium graphs they produced. In all thirteen test problems the algorithm proceeded monotonically through the family of economies from $\mathcal{E}(0)$ to $\mathcal{E}(1)$. No backtracking or "catastrophe" effects were observed. This may have happened because the economies in the test problems admit unique equilibria. On the other hand, the monotone behavior could have resulted from geometries similar to those illustrated in Figures 1.1.2 or 1.1.3. Unfortunately none of the known testable criteria for uniqueness apply to the economic models considered here because of the CRS production.

The case for uniqueness receives some support from the fact that in all problems for which comparisons are valid, the observed equilibria for $\mathcal{E}(0)$ and $\mathcal{E}(1)$ agree completely with those obtained by Scarf and Shoven using different computational techniques. The three six-commodity test problems reached an approximate equilibrium for $\mathcal{E}(0)$ identical to the one reported in Scarf's book [13], which was obtained via Scarf's algorithm. The five and fifteen-"commodity" examples (test problems 9, 10, and 13) matched Shoven's results at both ends of the equilibrium graph. Shoven obtained his pairs of equilibria by forcing the unevenly taxed economy to fit empirical observations and by subjecting the tax-equalized counterpart to a Newtonian search routine. Since the economic algorithm located the manually imposed equilibria from scratch and tied them to the Newtonian search results by equilibrium paths, it seems plausible that the equilibria in these models might be unique.

Price movements along the approximate equilibrium graphs were generally predictable considering the economic interpretation of the test problems. With rare exceptions prices moved monotonically through the *families of economies* reflecting, no doubt, the monotone nature of the deformations. The degree of price variation ranged from virtually nil to sixty-five percent. The most pronounced variations occurred in the four problems depicted in Figures 6.2.3 through 6.2.6. These figures attempt to convey the rudimentary geometry of each equilibrium graph by plotting individual price (and revenue) components on a single pair of axes. Logarithmic price scales are used to emphasize proportional changes.

The deformation in test problem 3 doubles the productivity of labor in all sectors. This effectively increases the *supply of labor available* to the economy. Consequently the prices of both types of labor (goods 2 and 3) fall off sharply during the course of the deformation. As the effective supply of labor increases, beginning capital (good 1) becomes relatively scarce as a factor of production and hence increases in price. Labor intensive outputs (good 4) decrease in price accordingly, while capital intensive items (good 5) become more expensive. Curiously during the initial third of the deformation, prices appear to move counter to the overall trends. The explanation for this phenomenon is hidden in the detailed production plans for the first few economies. As labor grows more efficient, technology in the non-durable goods sector shifts away from capital intensive activity 3 toward labor intensive activities 4

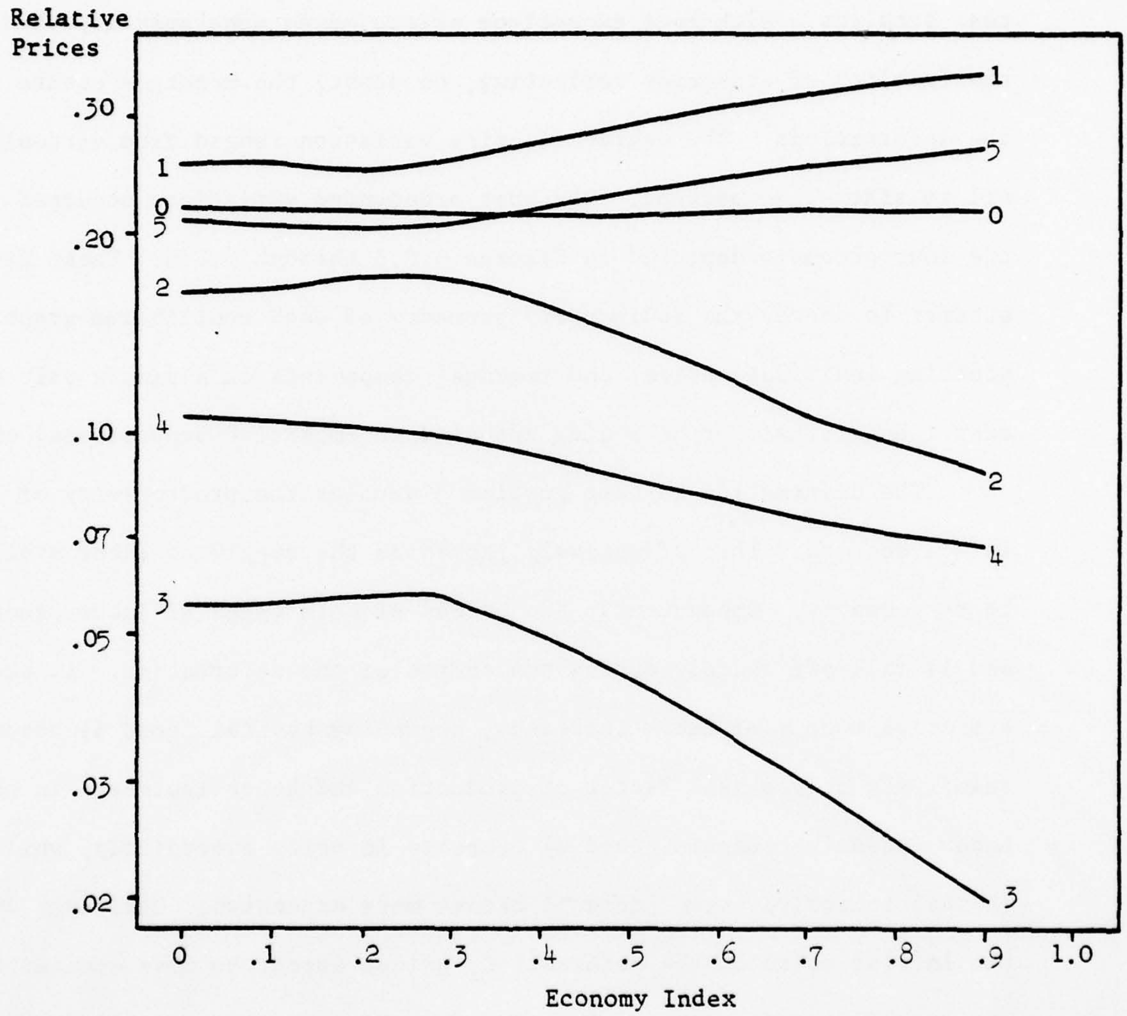

Figure 6.2.3. Price movements in test problem 3.

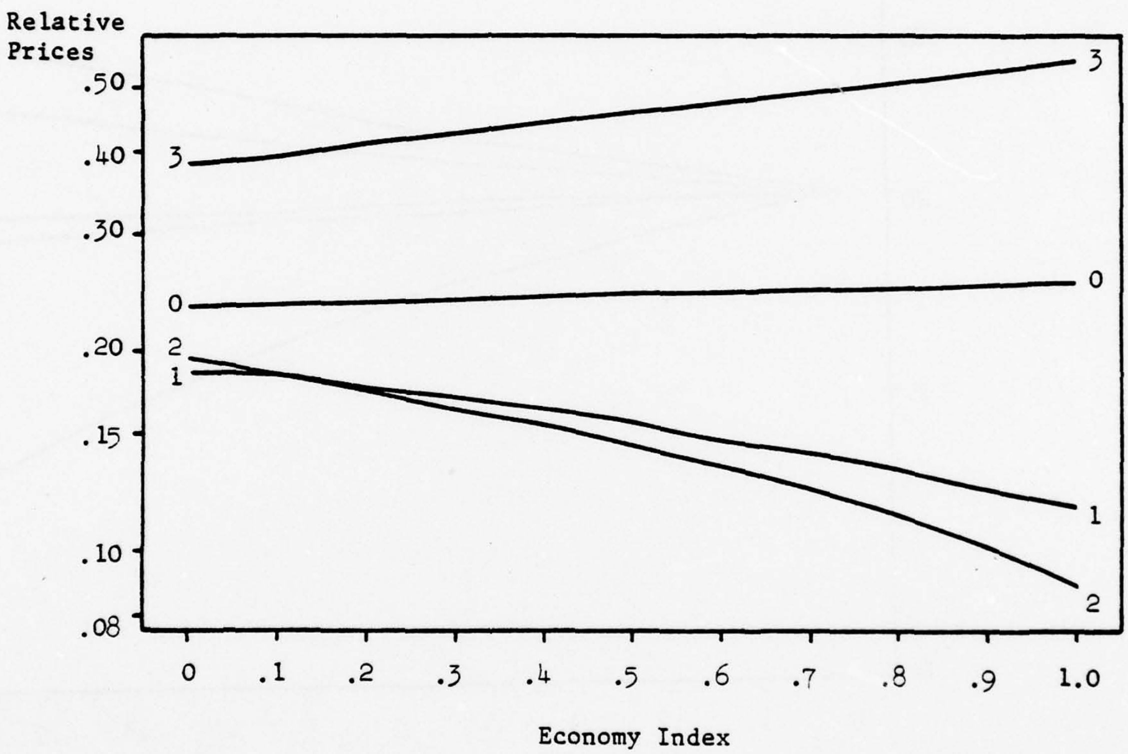

Figure 6.2.4. Price movements in test problem 7.

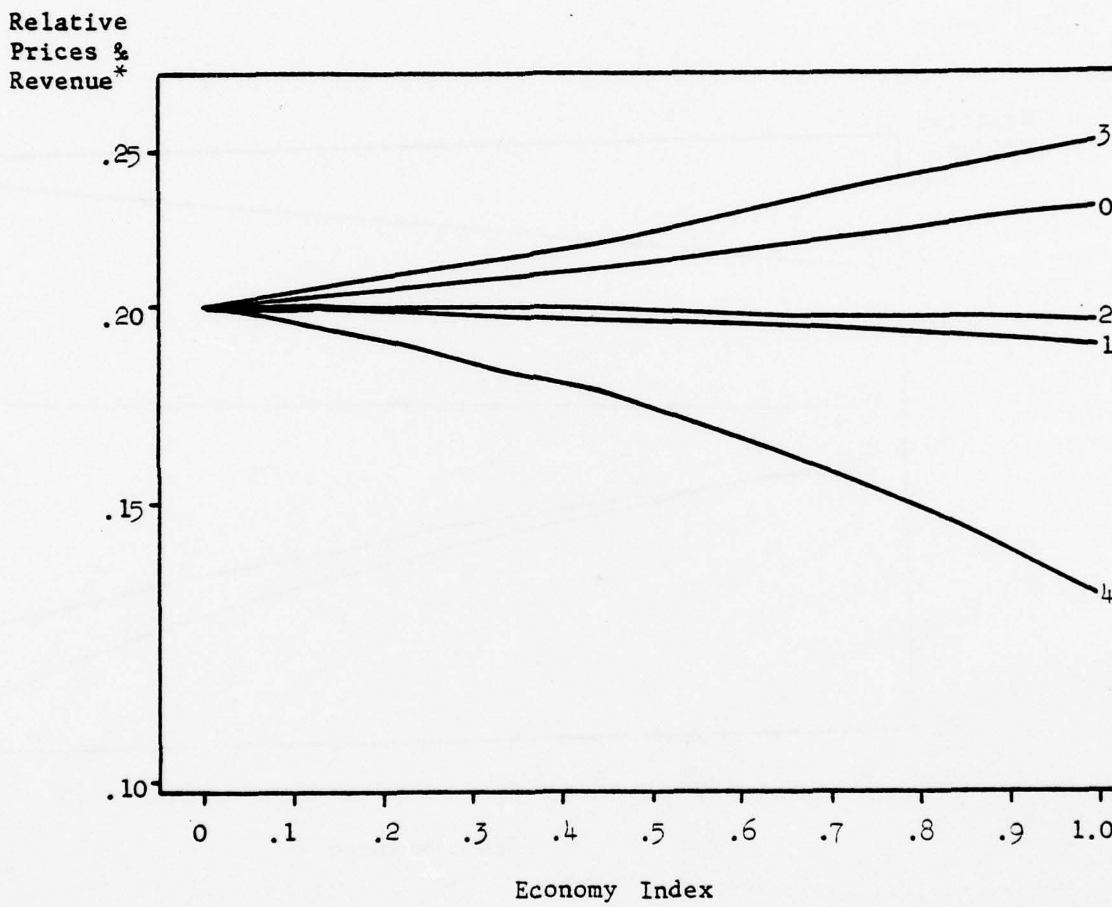

Figure 6.2.5. Price-revenue movements in test problem 9.

*Normalized so that $e(\pi, r) = 1$.

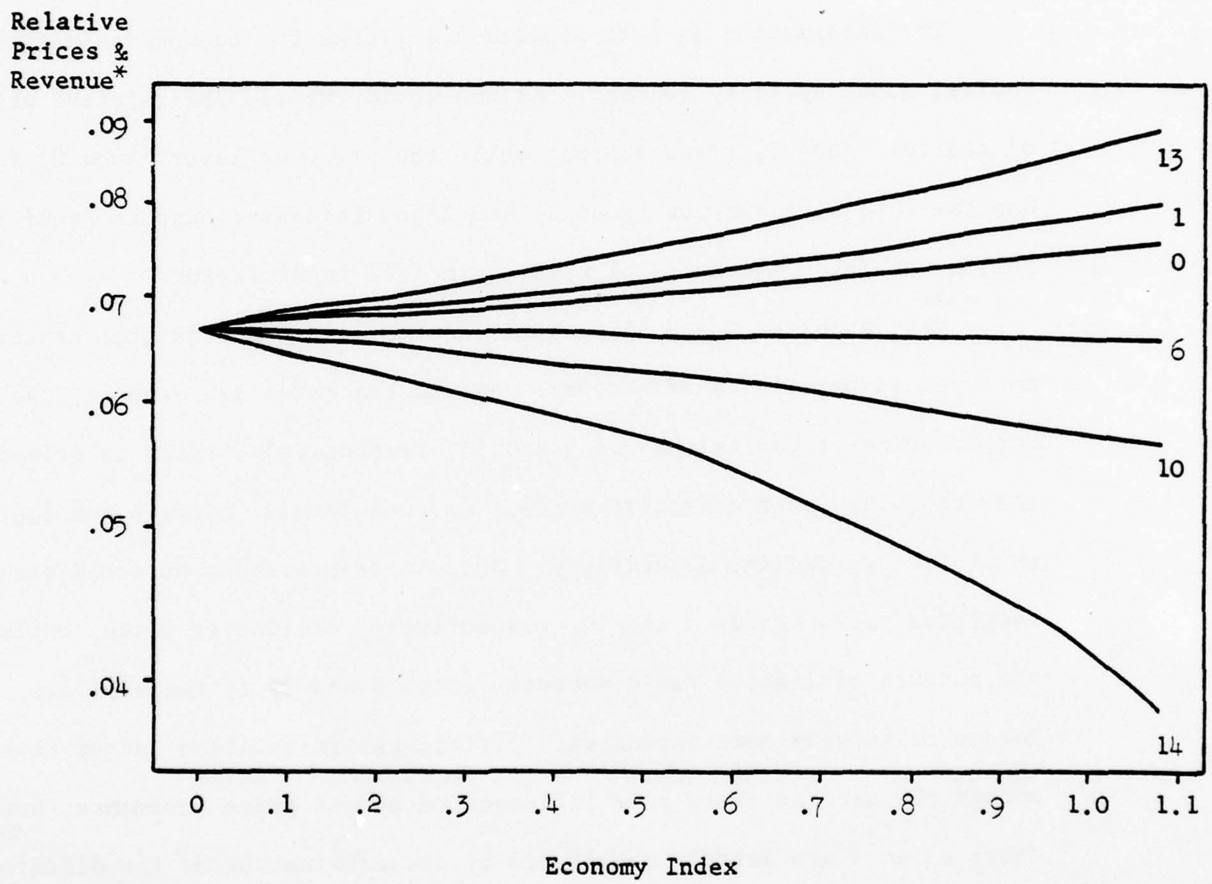

Figure 6.2.6. Selected price-revenue movements in test problem 13.

*Normalized so that $e(\pi, r) = 1$.

and 5. This shift absorbs the extra labor liberated by the deformation and then some. When the shift is complete the expected trends set in. The process resembles a change of basis in a parametric linear program and is responsible for the only non-monotone price movements observed in the test problems.

The deformation in test problem 7 depletes the economy's initial capital stock by fifty percent. As one would expect, the relative price of capital (good 3) rises sharply while the price of labor (good 2) falls. Capital intensive outputs (good 0) and labor intensive outputs (good 1) follow the price movements of their principal input factors.

Test problems 9 and 13 feature the removal of surtax from capital employed in heavily taxed sectors. As the tax rates are reduced, the market price of capital (goods 3 and 13, respectively) rises to offset its otherwise lower effective price. Revenue levels (goods 4 and 14, respectively) decline accordingly. Outputs from sectors burdened with heavy tax rates (goods 1 and 10, respectively) decline in price, while the outputs of lightly taxed sectors (goods 0 and 0, 1; respectively) become relatively more expensive. Differences in relative factor intensities across the sectors exert some influence on output price movements, but these effects are largely outweighed by the elimination of tax differentials.

6.3. Interpretations and Conclusions

Computational experience with the thirteen test problems indicates that the economic algorithm works in practice as well as in theory. The uniform approximation problem, a major obstacle to implementation, was successfully overcome by the dynamic manifold construction mechanism (at least for the class of examples considered here). The only disquieting aspect of the algorithm's performance is the massive amount of computational effort apparently required to solve even small examples. Various explanations were offered in the previous section for this unfortunate behavior. In the present section a more rigorous analysis of iteration counts will be conducted using theoretical lower bound formulae adapted from the work of Michael Todd.

In his paper on the design and critique of triangulations for computing fixed points [21], Todd introduces various theoretical measures of computational efficiency. One such measure is the directional density $N(G,d)$ of a triangulation G in the direction $d \in R^{n+1}$. $N(G,d)$ measures the average number of simplices in the triangulation G which intersect each unit length of a long straight line segment parallel to d. Todd derives a formula for $N(G,d)$ when G is the triangulation $J_1(\delta)$ defined in Section 4.2, namely

$$(6.3.1) \qquad N(J_1(\delta),\ d) = \frac{1}{\delta\|d\|_2} \sum_{i \leq j} \max\{|d(i)|,\ |d(j)|\} \quad .$$

Based on this formula the total number of simplices of $J_1(\delta)$ which meet a line segment in R^{n+1} (long compared to δ) with endpoints x_0 and x_1 can be accurately approximated by

$$(6.3.2) \qquad \frac{1}{\delta} \|x_1 - x_0\|_2 \, N(J_1(1), x_1 - x_0) \ .$$

Note that the directional density is "normalized" to a grid size of unity. This permits independent evaluation of the impact of grid size, segment length, and segment direction.

Expression 6.3.2 enables the calculation of theoretical lower bounds on the number of iterations required to generate approximate equilibrium graphs. First consider an equilibrium graph contained entirely in a manifold block of Type 0 (in D_2). Clearly the number of iterations required to generate such a graph must exceed the number of $(n+1)$-simplices lying along the line segment connecting its endpoints v_0 and v_1. This is precisely the number given by 6.3.2 with $x_0 = u^{-1}(v_0)$ and $x_1 = u^{-1}(v_1)$, where u is the affine homeomorphism defined in Section 4.2. Now suppose the equilibrium graph spans several manifold blocks of Type 0. Formula 6.3.2 still tells how many simplices intersect the line segment connecting the endpoints provided the $1/\delta$ factor is replaced by the average of such factors for the individual blocks, weighted by the thicknesses of the blocks. Finally consider an actual equilibrium graph generated by the algorithm. Such a graph typically passes through manifold blocks of all three types. Experience with the test problems indicates, however, that substantially all of the iterations take place in Type 0 blocks. Hence reasonable approximate lower bounds can be calculated by ignoring the transition layers and treating the manifold as if it consisted entirely of Type 0 blocks.

Straight line lower bounds for the thirteen test problems were calculated in the manner described above. The results of these calculations are displayed in Table 6.3.1, along with the ratios of actual iterations to the lower bounds. The table also contains a breakdown of the lower bounds into factors $1/\delta$, $\|x_1 - x_0\|_2$, and $N(J_1(1), x_1 - x_0)$. Decomposing iterations in this manner lends quantitative support to the informal analysis of iteration counts advanced in the previous section. Grid size, for instance, is clearly responsible for the overall order of magnitude of the counts, but does little to explain the differences among test problems. The normalized directional density $N(J_1(1), x_1 - x_0)$ accounts for most of the difference between the lower dimension problems and the fourteen and fifteen-"commodity" examples, but says nothing about problems of comparable size.

The most telling source of variation in iteration counts among similarly sized problems is the length of the transformed line segment $\|x_1 - x_0\|_2$. (Contrast the values of this factor for problems 3 and 7 with those for problems 4 and 6.) Differences in $\|x_1 - x_0\|_2$ are in turn caused by different degrees of price-revenue variation along the approximate equilibrium graphs.* By attributing the degree of price change to the "severity" of the deformation, one obtains a useful heuristic

*The reasoning behind this assertion is as follows. All test problems (except number 12) ran substantially to completion. The economy index scaling factors were essentially the same. Hence all problems covered the same vertical distance in $S \times [0, \infty)$. Mapping back to $R^n \times [0,\infty)$ via u^{-1} preserves vertical displacements. Hence all variations in $\|x_1 - x_0\|_2$ were induced by horizontal displacements, which are in turn caused by price-revenue changes.

TABLE 6.3.1. Comparison of Actual Computational Effort With Theoretical Lower Bounds

Test Problem	Actual # Iterations Between Endpoints	Straight Line Lower Bound On Iterations	Reciprocal* of Average Grid Size ($\times 10^4$)	Length of Transformed Line Segment ($\times 10^{-2}$)	Normalized Directional Density	Ratio of Actual to Lower Bound
1	64,009	40,941	6.67	6.76	9.09	1.56
2	17,050	9,936	7.69	1.37	9.46	1.72
3	154,924	48,471	5.00	11.03	8.79	3.20
4	4,270	2,391	5.00	0.91	5.26	1.79
5	21,539	5,684	4.00	2.78	5.11	3.79
6	4,240	1,572	4.35	0.80	4.52	2.70
7	58,274	50,004	5.88	17.35	4.90	1.17
8	11,366	3,488	1.15	4.77	6.36	3.26
9	15,028	5,518	1.00	7.83	7.05	2.72
10	15,229	5,190	0.83	8.92	6.98	2.93
11	19,580	5,489	2.86	0.84	22.83	3.57
12	31,181	5,803	2.00	1.27	22.91	5.37
13	192,587	32,163	1.12	8.19	34.95	5.99

*Surrogate for average reciprocal grid size. Causes lower bounds to be consistently understated, potentially by tens of percent.

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in terms of which the amount of computational effort required in particular examples can be explained and possibly anticipated.

For purposes of evaluating the efficiency of the algorithm, the most important figures in Table 6.3.1 are the ratios of actual iterations to straight line lower bounds. These ratios provide a rough measure of the degree of oscillation experienced by the algorithm as it progressed through the families of economies. Considering that the lower bounds do not allow for the natural curvature of the equilibrium graphs, and that they are uniformly understated (see footnote to Table 6.3.1), the ratios indicate very respectable performance on the part of the algorithm. In some instances the lower bounds came close to being achieved. At the very worst the discrepancy was half an order of magnitude. Contrast this to the four orders of magnitude contributed by reciprocal grid size. Also, when evaluating the algorithm's performance, it should be kept in mind that the labeling used in the algorithm is extremely discontinuous, a trait which tends to encourage oscillation in search algorithms of this type.

On the basis of the ratio analysis it seems reasonable to conclude that the exorbitant computational effort expended in computing approximate equilibrium graphs results not from the inefficiency of the economic algorithm, but rather from the vast amount of information inherent in precisely specified equilibrium paths. Granted that parametric equilibrium problems are intrinsically expensive to solve, some means must be found to conserve iterations if the algorithm is ever to be applied beyond small research problems. Two possibilities come to mind. One is to reduce the dimension of the price-revenue simplex over which the search for equilibria takes place. Various dimension reduction techniques that exploit special structure in the economic model have been developed for use with Scarf's algorithm. Some of these techniques can undoubtably be adapted to parametric calculations.

The other possibility is to expand the mesh of the grid on which the algorithm operates. Ideally this could be accomplished without sacrificing approximation accuracy. Unfortunately all attempts to improve the range error/domain error relationship proved unsuccessful, probably because of labeling discontinuities. The only recourse, therefore, appears to be relaxing accuracy requirements. The numerical experiments may in fact have demanded too much of the algorithm by requiring that entire paths of equilibria be computed with the precision one normally expects for single equilibria. Once the general shape of an equilibrium graph is determined, highly refined approximations can easily be computed for particular economies of interest. Of course one never knows how far the tolerances can be relaxed and still maintain adequate resolution. This can only be determined by trial and error for particular examples. Nevertheless, the potential for acceleration is so vast (ranging up to three orders of magnitude before demand labels disappear) that such experimentation warrants serious consideration.

APPENDIX A

TECHNICAL LEMMAS

This appendix develops a portion of the mathematical machinery needed for the proofs of Chapter 3. The machinery is designed to cope with various abstractions built into the economic model, e.g., generalized tax functions (Lemma A.1), uncountable activity sets (Lemmas A.2 and A.3), and multivalued demand correspondences (Lemma A.5). The results are readily stated and proved in terms independent of the formalism of Chapter 3, and hence have been removed to this appendix to streamline the exposition.

A.1. LEMMA. Let $\Phi : D \to (R^m)^*$ be a bounded u.s.c. correspondence on a compact domain $D \subset R^k$. Then for any closed subset E of D, $\bigcup_{x \in E} \Phi(x)$ is compact.

Proof: Since Φ is u.s.c. and bounded and D is compact, the graph of Φ is a compact subset of $R^k \times R^m$. The set $\bigcup_{x \in E} \Phi(x)$ is precisely the projection onto R^m of the compact set formed by intersecting the graph of Φ with the closed cylinder $E \times R^m$, and is therefore compact. □

A.2. LEMMA. Let $A^k \in R^{m \times n}$ and $b^k \in R^m$ for $0 \leq k \leq \infty$. Define $P^k = \{x^k \in R^n_+ : A^k x^k \geq b^k\}$. Suppose $A^k \to A^\infty$, $b^k \to b^\infty$, and P^∞ is non-empty and bounded. Then $\exists\, N \in Z_+$ s.t. $\bigcup_{k=N}^{\infty} P^k$ is bounded.

Proof: Since P^{∞} is non-empty and bounded the system $A^{\infty} x^{\infty} \geq 0$, $x^{\infty} > 0$ must be infeasible. According to Tucker's alternative theorem [11] $\exists\, y \in R^m_+$ s.t. $yA^{\infty} \ll 0$. Clearly $yA^k \to yA^{\infty}$, and since $b^k \to b^{\infty}$, $\exists\, N \in Z_+$ s.t. $k \geq N$ implies $yA^k \leq \frac{1}{2}\, yA^{\infty} \ll 0$ and $b^k \geq b^{\infty}-e$. Thus for $k \geq N$ and x^k in P^k,

$$\frac{1}{2}\, yA^{\infty} x^k \geq yA^k x^k \geq yb^k \geq y(b^{\infty}-e) ,$$

which implies

$$x^k(i) \leq \frac{2y(b^{\infty}-e)}{(yA^{\infty})(i)}$$

for $0 \leq i \leq n-1$. Hence $\bigcup\limits_{k=N}^{\infty} P^k$ is bounded. □

A.3. LEMMA. Let C and $\mathcal{D}$ be non-empty compact subsets of R^m. Define

$$\mathcal{C}^n = \{C \in R^{m \times n} : \text{columns of } C \text{ lie in } C\} .$$

For C in $\mathcal{C}^n$ and d in $\mathcal{D}$ define

$$P(C,d) = \{x \in R^n_+ : Cx \geq d\} .$$

Suppose $P(C,d)$ is non-empty and bounded for each (C,d) in $\mathcal{C}^n \times \mathcal{D}$. Then

$$\bigcup_{(C,d) \in \mathcal{C}^n \times \mathcal{D}} P(C,d) \quad \text{is bounded.}$$

<u>Proof</u>: If the union is unbounded, then there exists a sequence $\langle x^k, C^k, d^k \rangle$ in $R^n_+ \times \mathcal{C}^n \times \mathcal{D}$ such that $x^k \in P(C^k, d^k)$ and $\|x^k\| \to \infty$. Since $\mathcal{C}^n \times \mathcal{D}$ is compact there exists a subsequence of $\langle x^k, C^k, d^k \rangle$, for convenience also indexed by k, along which $C^k \to C^\infty \in \mathcal{C}^n$ and $d^k \to d^\infty \in \mathcal{D}$. By Lemma A.2 $\exists\, N \in Z_+$ s.t. $\bigcup_{k=N}^{\infty} P(C^k, d^k)$ is bounded, contradicting the fact that $\|x^k\| \to \infty$. $\square$

A.4. LEMMA. Let X be a compact subset of R^n, and let A and B be closed subsets of X. Suppose $\langle x^k \rangle$ is a sequence in X such that as $k \to \infty$, $\|x^k - x^{k+1}\| \to 0$, $\mathrm{dist}(x^k, A) \to 0$, and $\mathrm{dist}(x^k, B) \to 0$. Then the set Λ of limit points of $\langle x^k \rangle$ is a connected subset of X which meets both A and B.

<u>Proof</u>: Since X is compact, Λ is non-empty and compact. Suppose Λ is not connected. Then there exist open subsets U and V of X which are disjoint and which meet and partition Λ. Both $\Lambda \cap U$ and $\Lambda \cap V$ are compact since they are each complements in Λ of relatively open subsets of Λ (namely each other). Hence they are a positive distance apart. Let M and N be open neighborhoods of $\Lambda \cap U$ and $\Lambda \cap V$ which are also a positive distance apart. Since $\langle x^k \rangle$ meets both M and N infinitely often and since $\|x^k - x^{k+1}\| \to 0$ as $k \to \infty$, the sequence

$\langle x^k \rangle$ must meet the compact set $X \backslash (M \cup N)$ infinitely often and hence have a limit point there, contradicting the fact that $\Lambda \subset M \cup N$.

Since A and B are closed, for each k in Z_+ there exist closest points a^k and b^k to x^k in A and B respectively. Since A and B are compact the sequences $\langle a^k \rangle$ and $\langle b^k \rangle$ have limit points a^∞ in A and b^∞ in B. Clearly $\mathrm{dist}(a^\infty, \langle x^k \rangle)$ $= \mathrm{dist}(b^\infty, \langle x^k \rangle) = 0$, so both a^∞ and b^∞ belong to Λ. $\square$

A.5. LEMMA. Let $\Phi : D \to (R^m)^*$ be a bounded continuous correspondence on a compact domain $D \subset R^n$. Assume that R^n and R^m are normed by $\| \|_p$ and $\| \|_q$ respectively. Then $\forall \epsilon > 0$ there exists $\delta > 0$ such that for all x,y in D with $\|x-y\|_p < \delta$, and a in $\Phi(x)$, there exists b in $\Phi(y)$ such that $\|a-b\|_q < \epsilon$.

<u>Proof</u>: If the conclusion were false, then there would exist $\epsilon > 0$ s.t. $\forall \delta > 0\ \exists\ x,y$ in D and a in $\Phi(x)$ s.t. $\|x-y\|_p < \delta$ and $\|a-b\|_q \geq \epsilon$ for all b in $\Phi(y)$. Since D is compact and $\Phi(D)$ is bounded, one could then construct a sequence $\langle x^k, y^k, a^k, \delta^k \rangle$ along which $\delta^k \to 0$, $\|x^k - y^k\|_p < \delta^k$, $x^k \to x^\infty \in D$, $a^k \in \Phi(x^k)$, $a^k \to a^\infty \in R^m$, and $\forall\, b^k$ in $\Phi(y^k)$, $\|a^k - b^k\|_q \geq \epsilon$. Clearly $y^k \to x^\infty$, and by the u.s.c. of Φ, $a^\infty \in \Phi(x^\infty)$. But for large k, $\|a^k - a^\infty\|_q < \frac{\epsilon}{2}$, and thus $\forall\, b^k$ in $\Phi(y^k)$, $\|b^k - a^\infty\|_q \geq \frac{\epsilon}{2}$. This contradicts the l.s.c. of Φ. $\square$

A.6. DEFINITION. The constant $\delta > 0$ corresponding to a given $\epsilon > 0$ in Lemma A.5 is said to be a p-q uniformity constant for (Φ, ϵ).

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A philosophical problem arises when one attempts to predict a competitive economy's response to a fundamental change in its structure with the aid of a competitive equilibrium model. Unless the model is known to admit unique solutions, there is little basis for assuming that the computed equilibrium will be attained, even though the model accurately describes the economy's structure and the behavior of its agents. If, however, one is able to arrive at the new model by continuously deforming the old one, then the two versions generally admit solutions which are connected by a path of equilibria arising

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by Charles R. Engles

from the continuum of intermediate economies. By ascribing a suitable dynamic interpretation to the deformation, one obtains a rationale for expecting the path-connected solutions to be mutually attained.

The description of economic deformations and the computation of equilibrium paths is the central theme of this study. A general mathematical framework for modeling economies under deformation is developed by expanding Herbert Scarf's original activity analysis formulation to include uncountable unit activity sets, unbounded multi-valued demand correspondences, and tax and revenue systems similar to those introduced by John Shoven and John Whalley. Deformations of virtually all economic constructs are allowed in this general model.

The computation of equilibrium paths is accomplished by a simplicial pivot algorithm designed along the lines of the homotopy-type fixed point techniques pioneered by Curtis Eaves. The dimension normally used to refine piecewise linear approximations now serves as the index of the economic deformation. To make this approach viable in practice, a new family of triangulations of Euclidean space is fashioned out of two conventional triangulations invented by Michael Todd. The geometry of these triangulations can be dynamically altered by the algorithm as it attempts to maintain uniform approximation error along the equilibrium path.

The economic model and computational algorithm are translated into a set of computer routines which generate explicit numerical approximations to equilibrium paths for a variety of examples. Due to the vast amount of information embodied in an equilibrium path, problems of this type require a great deal of computational effort. A detailed analysis of the behavior of the algorithm on a series of test problems is presented in the final chapter.